Deformation Theory of Compact Complex Manifolds and CR Manifolds

NG Wai Man

A Thesis Submitted in Partial Fulfillment
of the Requirements for the Degree of
Master of Philosophy
in
Mathematics

June 2006

Thesis/Assessment Committee

Professor TAM, Luen Fai (Chair)
Professor LUK, Hing Sun (Thesis Supervisor)
Professor CHOU, Kai Sing (Committee Member)
Professor YAU, Shing Toung, Stephen (External Examiner)

Abstract

In this thesis we are going to study the deformations of compact complex manifolds and pseudoconvex CR manifolds. The existence and the completeness theorems for complex analytic family of compact complex manifolds, originally worked out by Kodaira and Spencer [8], will be proved by applying the analysis of the complex Laplacian. Then, we will study partially complex manifolds, in particular CR manifolds, together with their $\overline{\partial}$ complexes. The upper semi-continuity theorems for both families of complex manifolds and CR manifolds will be proved to serve as a comparison of the elliptic and subelliptic estimates.

摘要

在本論文中我們將討論緊緻複流形以及擬凸柯西黎曼(CR)流形的變形。透過對複調和算子的分析，我們將證明關於複流形複解析族上的存在性及完備性定理。該等定理最先為小平邦彥及史賓塞[8] 所證明。然後，我們將研究半複流形、特別是CR流形及其$\bar{\partial}$ 復形。藉著證明複流形及CR流形族的上半連續定理，我們將可比較橢圓算子及劣橢圓算子的推估。

ACKNOWLEDGMENTS

I would like to express my sincere gratitude to my supervisor, Professor H.S. Luk for his supervision and guidance during my two-year postgraduate study. I benefit a lot by learning from him, and without his advice and assistance, I would not be able to stride across those difficult days, and this thesis would not be able to appear. I would also like to thank the teachers in the mathematics department, for they have brought me various important insights in the realm of mathematics throughout these years.

Contents

Chapter 1

Introduction

In this thesis we are going to study the deformations of compact complex manifolds and of compact strongly pseudoconvex CR manifolds. We first introduce the concept of a differentiable family of compact complex manifolds, in which the complex structures vary only differentiably with the given parameters. Such variation can be measured by the infinitesimal deformation, studied in **Chapter 2**, with the help of sheaf language. This measurement should serve as a derivative for the family with respect to the parameter. Thus, a zero infinitesimal deformation should imply a locally trivial family, or in other words, a family which is locally a product of a fixed manifold M_0 with a subdomain of the parameter space. Unfortunately, such implication does not hold in general, and Kodaira and Spencer [8] managed to establish this result for complex analytic families under an extra assumption on the dimension of $H^1(M_t, \Theta_t)$.

Chapter 3 aims at the following existence theorem of a complex analytic family: Given a compact manifold M and an element $\theta \in H^1(M, \Theta)$, we want to construct a family $\mathcal{M}$ containing M and whose infinitesimal deformation at 0 is θ. The result that, once such a family is established, the Lie bracket $[\dot{\theta}(0), \dot{\theta}(0)]$ must vanish in $H^2(M, \Theta)$, gives a hint that the construction of the family can be made possible by placing suitable assumption(s) on $H^2(M, \Theta)$. In the paper of Kodaira and Spencer [8], they managed to do so by assuming $H^2(M, \Theta) = 0$, and applying the Newlander-Nirenberg theorem [12] to the resulting integrable almost complex structures.

One central problem in this deformation theory is to determine all local deformations of complex structures associated to a given compact complex manifold M. To address this question properly, one looks for a complete family containing the manifold in question, by which one means that locally the family should contain all possible deformations of the given manifold. We will prove a completeness theorem which gives a sufficient condition for a complex analytic family to be complete. Then, the concepts of effective parameters and moduli are introduced. Both of them are related to the quantity $\dim H^1(M, \Theta)$, and together they give a count of all local deformations of complex structures on compact complex manifolds. All these are the main topics of **Chapter 4**.

In **Chapter 5** we focus on partially complex manifolds, and in particular on CR manifolds. We first introduce the extrinsic notion of partially complex manifolds and their tangential complexes. Then we define abstract CR manifolds and build up a cohomology theory from their $\overline{\partial}$ complexes. Further, we introduce the Levi form and restrict to strongly pseudoconvex CR manifolds, where we can develop a theory of harmonic forms. Differentiable families of compact strongly pseudoconvex CR manifolds will then be studied, and a corresponding upper semi-continuity theorem will be proved in **Chapter 6**, in parallel with a similar result for the compact complex manifolds.

In **Chapter 6** we study the basic properties of the Laplacians on complex and CR manifolds, and illustrate the applications of harmonic theory to some stability theorems. Roughly speaking, the two Laplacians on complex manifolds and on CR manifolds are rather different: one is strongly elliptic and the other is subelliptic. To better streamline our discussion, some of the arguments are given in the Appendices.

Chapter 2

Infinitesimal Deformations for Compact Complex Manifolds

In this chapter we first define the infinitesimal deformation for any differentiable family of compact complex manifolds, first for real parameters, then for complex ones. Intuitively this infinitesimal deformation is a derivative of the family. Our approach follows closely the one suggested by Kodaira [6],[7],[8].

2.1 Differentiable Family

Definition 2.1 *Let $B \subseteq \mathbb{R}^m$ be a domain, and M_t^n be a compact complex manifold for each $t \in B$, then $\{M_t \mid t \in B\}$ is a differentiable family of compact complex manifolds if there exists a differentiable manifold $\mathcal{M}$ and a C^∞ onto map $\varpi : \mathcal{M} \to B$ such that*

1. *The rank of the Jacobian of ϖ is m at each point of $\mathcal{M}$.*

2. *$\varpi^{-1}(t) = M_t$ is a compact connected subset of $\mathcal{M}$ for each $t \in B$.*

3. *There exists a locally finite open covering $\{\mathcal{U}_j | j = 1, 2, ...\}$ of $\mathcal{M}$ and C^∞*

functions $z_j^1(p), ..., z_j^n(p), j = 1, 2, ...$ *defined on* $\mathcal{U}_j$ *such that for each* t

$$\{p \to (z_j^1(p), ..., z_j^n(p)) \mid \mathcal{U}_j \cap \varpi^{-1}(t) \neq \emptyset\}$$

forms a system of local complex coordinates of M_t.

In this situation, $t \in B$ is called the parameter of the differentiable family, and B the parameter space. Under the setting in **Definition 1.1**, we can derive formulas in local coordinates as follows.

The local coordinate charts are $x_j : \mathcal{M} \to \mathbb{C}^n \times B$ with

$$x_j(p) = (z_j^1(p), ..., z_j^n(p), t^1, ..., t^m) \quad , \quad \forall p \in \mathcal{M} \cap \mathcal{U}_j \ , \ \varpi(p) = (t^1, ..., t^m)$$

whose transition functions are $f_{jk} = (f_{jk}^1, ..., f_{jk}^n) : z_k(\mathcal{U}_j \cap \mathcal{U}_k) \to z_j(\mathcal{U}_j \cap \mathcal{U}_k)$ such that $f_{jk}^\alpha, 1 \leq \alpha \leq n$ is C^∞ in each variable, and is holomorphic in $z_k^1, ..., z_k^n$ for each fixed t, and

$$z_j = f_{jk}(z_k, t), \qquad \text{whenever} \qquad \mathcal{U}_j \cap \mathcal{U}_k \neq \emptyset \tag{2.1}$$

$$f_{ik}(z_k, t) = f_{ij}(f_{jk}(z_k, t), t), \qquad \text{whenever} \qquad \mathcal{U}_j \cap \mathcal{U}_j \cap \mathcal{U}_k \neq \emptyset \tag{2.2}$$

Without loss of generality, we can choose x_j so that $x_j(\mathcal{U}_j) = U_j \times I_j$, where U_j is a polydisc in $\mathbb{C}^n$ and I_j is a multi-interval in B. Then by identifying p with $x_j(p)$, we have

$$M_t = \varpi^{-1}(t) = \cup_{I_j \ni t}(U_j \times t) \cong \cup_{I_j \ni t} U_j$$

We now construct the infinitesimal deformation for any given differentiable family. We first consider the case where $t \in \mathbb{R}$ (i.e. $m = 1$). Here we assume $x_j(\mathcal{U}_j) = U_j \times I_j$

$$\begin{aligned} f_{ik}^\alpha(z_k, t) &= f_{ij}^\alpha(f_{jk}^1(z_k, t), ..., f_{jk}^n(z_k, t), t) \\ \Rightarrow \frac{\partial f_{ik}^\alpha}{\partial t}(z_k, t) &= \frac{\partial f_{ij}^\alpha(z_j, t)}{\partial t} + \sum_{\beta=1}^n \frac{\partial z_i^\alpha}{\partial z_j^\beta} \frac{\partial f_{jk}^\beta(z_k, t)}{\partial t} \\ \Rightarrow \sum_{\alpha=1}^n \frac{\partial f_{ik}^\alpha}{\partial t}(z_k, t) \frac{\partial}{\partial z_i^\alpha} &= \sum_{\alpha=1}^n \frac{\partial f_{ij}^\alpha}{\partial t}(z_j, t) \frac{\partial}{\partial z_i^\alpha} + \sum_{\beta=1}^n \frac{\partial f_{jk}^\beta(z_k, t)}{\partial t} \frac{\partial}{\partial z_j^\beta} \end{aligned}$$

Introducing a holomorphic vector field $\theta_{jk}(t)$ on $U_j \cap U_k \neq \emptyset$, where $M_t = \cup_j U_j$,

$$\theta_{jk}(t) = \sum_{\alpha=1}^{n} \frac{\partial f_{jk}^{\alpha}}{\partial t}(z_k, t)\frac{\partial}{\partial z_j^{\alpha}}, \qquad z_k = f_{jk}(z_j, t) \tag{2.3}$$

Then it is easy to observe that

$$\theta_{jk}(t) - \theta_{ik}(t) + \theta_{ij}(t) = 0 \qquad \text{where} \qquad U_i \cap U_j \cap U_k \neq \emptyset \tag{2.4}$$

$$\theta_{kj}(t) = -\theta_{jk}(t) \qquad \text{where} \qquad U_i \cap U_j \neq \emptyset \tag{2.5}$$

The equalities (2.4) and (2.5) imply that $\{\theta_{jk}(t)\} \in Z^1(\mathfrak{U}_t, \Theta_t)$, where $\mathfrak{U}_t = \{U_j\}$ and Θ_t is the sheaf of germs of holomorphic vector fields over M_t. By the canonical inclusion $H^1(\mathfrak{U}_t, \Theta_t) \subseteq H^1(M_t, \Theta_t)$, $\{\theta_{jk}(t)\}$ determines an element $\theta(t)$ in $H^1(M_t, \Theta_t)$:

Theorem 2.1 *The element $\theta(t) \in H^1(M_t, \Theta_t)$ is uniquely determined by $\{\theta_{jk}(t)\}$. We call $\theta(t)$* ***the infinitesimal deformation of*** *M_t, usually denoted as $\frac{dM_t}{dt}$.*

PROOF OF THEOREM 2.1

We need to verify the following 3 facts:

- *The construction works without the assumption $x_j(\mathcal{U}_j) = U_j \times I_j$*: For such $\mathcal{U}_j$, we consider a new covering $\mathfrak{A} = \{U_{jt}\}, \mathcal{U}_j \cap M_t = U_{jt} \times t$. Then the above formality can be repeated on $\mathfrak{A}$.

- *$\theta(t)$ does not change under any refinement of $\mathfrak{U}_j = \{\mathcal{U}_j\}$*: This is because the resulting $\{\widetilde{\theta_{jk}(t)}\}$ defined on the refinement is simply the restriction of the original cocycles.

- *$\theta(t)$ does not change under any choice of local coordinates.*

For the last case, it suffices to show that given $x_j = (z_j, t)$ and $u_j = (w_j, t)$ on each U_j, $\eta(t)$ defined by $\{u_j\}$ coincides with $\theta(t)$ defined by $\{x_j\}$.

Suppose $w_j = h_{jk}(w_k, t)$ on $\mathcal{U}_j \cap \mathcal{U}_k \neq \emptyset$, then

$$\eta_{jk}(t) = \sum_{\alpha=1}^{n} \frac{\partial h_{jk}^{\alpha}(w_k,t)}{\partial t}\frac{\partial}{\partial w_j^{\alpha}}, \qquad w_k = h_{kj}(w_j,t)$$

Write $w_j^{\alpha} = g_j^{\alpha}(z_j^1, ..., z_j^n, t)$, for some g_j^{α} holomorphic in $z_j^1, ..., z_j^n$, then

$$\begin{aligned}
g_j^{\alpha}(f_{jk}(z_k,t),t) &= g_j^{\alpha}(z_j,t) = w_j^{\alpha} = h_{jk}^{\alpha}(w_k,t)\\
&= h_{jk}^{\alpha}(g_k(z_k,t),t)\\
\Rightarrow \quad \sum_{\beta} \frac{\partial g_j^{\alpha}}{\partial z_j^{\beta}}\frac{\partial f_{jk}^{\beta}}{\partial t} + \frac{\partial g_j^{\alpha}}{\partial t} &= \sum_{\beta} \frac{\partial h_{jk}^{\alpha}}{\partial w_k^{\beta}}\frac{\partial g_k^{\beta}}{\partial t} + \frac{\partial h_{jk}^{\alpha}}{\partial t}\\
\Rightarrow \quad \sum_{\beta} \frac{\partial f_{jk}^{\beta}}{\partial t}\frac{\partial w_j^{\alpha}}{\partial z_j^{\beta}} + \frac{\partial g_j^{\alpha}}{\partial t} &= \sum_{\beta} \frac{\partial g_k^{\beta}}{\partial t}\frac{\partial w_j^{\alpha}}{\partial w_k^{\beta}} + \frac{\partial h_{jk}^{\alpha}}{\partial t}\\
\Rightarrow \quad \sum_{\beta} \frac{\partial f_{jk}^{\beta}}{\partial t}\frac{\partial}{\partial z_j^{\beta}} + \sum_{\alpha} \frac{\partial g_j^{\alpha}}{\partial t}\frac{\partial}{\partial w_j^{\alpha}} &= \sum_{\beta} \frac{\partial g_k^{\beta}}{\partial t}\frac{\partial}{\partial w_k^{\beta}} + \sum_{\alpha} \frac{\partial h_{jk}^{\alpha}}{\partial t}\frac{\partial}{\partial w_j^{\alpha}}\\
\Rightarrow \quad \theta_{jk}(t) + \theta_j(t) &= \theta_k(t) + \eta_{jk}(t),
\end{aligned}$$

where $\theta_j(t) := \sum_{\alpha} \frac{\partial g_j^{\alpha}}{\partial t}\frac{\partial}{\partial w_j^{\alpha}}$. This implies that $\{\theta_{jk}(t)\} - \{\eta_{jk}(t)\} = \delta\{\theta_j(t)\}$ in $C^0(\mathfrak{U}_t, \Theta_t)$, where $\mathfrak{U}_t = \{\mathcal{U}_j \cap M_t\}$ and hence we are done. ■

In general, when $B \subseteq \mathbb{R}^m, m \geq 1$,we can define the infinitesimal deformation in a similar manner: $\forall \frac{\partial}{\partial t} = \sum_{\lambda=1}^{m} c_{\lambda}\frac{\partial}{\partial t_{\lambda}} \in T_t(B)$, define

$$\theta_{jk}(t) = \sum_{\alpha=1}^{n} \frac{\partial f_{jk}^{\alpha}(z_k, t_1, ..., t_m)}{\partial t}\frac{\partial}{\partial z_j^{\alpha}}, \qquad z_k = f_{kj}(z_j,t)$$

This in turns defines an $\mathbb{R}$-linear map $\rho_t : T_t(B) \to H^1(M_t, \Theta_t)$, where $\rho_t\Big(\frac{\partial}{\partial t}\Big) = \frac{\partial M_t}{\partial t}$.

2.2 Local Triviality

Definition 2.2 *Given two differentiable families $(\mathcal{M}, B, \varpi)$ and $(\mathcal{N}, B, \pi)$ on the same $B \subseteq \mathbb{R}^n$, $\mathcal{M}$ and $\mathcal{N}$ are equivalent if there is a diffeomorphism $\Phi : \mathcal{M} \to \mathcal{N}$*

such that Φ is a biholomorphism between $M_t = \varpi^{-1}(t)$ and $N_t = \pi^{-1}(t)$ for each $t \in B$.

REMARK If $\mathcal{M}$ and $\mathcal{N}$ are equivalent, then we can consider $\mathcal{M}$ as the same differentiable family with $\mathcal{N}$ by identifying $p \in \mathcal{M}$ with $\Phi(p) \in \mathcal{N}$.

Definition 2.3 *1. $(\mathcal{M}, B, \varpi)$ is **trivial** if it is equivalent to $(M \times B, B, \pi)$ where $M = \varpi^{-1}(t^0)$ for some $t^0 \in B$.*

*2. $(\mathcal{M}, B, \varpi)$ is **locally trivial** if for each $t \in B$, there is a subdomain $I \subseteq B$ containing t such that $(\mathcal{M}_I, I, \varpi)$ is a trivial differentiable family.*

REMARK If $(\mathcal{M}, B, \varpi)$ is locally trivial, then each $M_t = \varpi^{-1}(t)$ is biholomorphically equivalent to a fixed $M = \varpi^{-1}(t^0)$. Conversely (a result from Fischer and Grauert), if each M_t is biholomorphic to a fixed M, then $(\mathcal{M}, B, \varpi)$ is locally trivial.

Theorem 2.2 *If $(\mathcal{M}, B, \varpi)$ is locally trivial, then $\theta(t) = 0$.*

PROOF OF THEOREM 2.2

This is because the transition functions for $\mathcal{M}$ within a subdomain I of B are independent of the parameter t. ■

It turns out that a zero infinitesimal deformation reflects the local triviality of $(\mathcal{M}, B, \varpi)$ under some conditions on the dimension of $H^1(M_t, \Theta_t)$.

Lemma 2.1 *Suppose $\theta(t) = 0$ on the differentiable family $(\mathcal{M}, B, \varpi)$, then we can define a vector field on $\mathcal{M}_I = \bigcup_{j=1}^{l} U_j \times I$.*

PROOF OF LEMMA 2.1

Write $\theta_{jk}(t) = \sum_{\alpha=1}^{n} \frac{\partial f_{jk}^\alpha}{\partial t} \frac{\partial}{\partial z_j^\alpha}$, whose cohomology class is $\theta(t)$. By the hypothesis, there exists $\{\theta_j(t)\} \in C^0(\mathfrak{U}_t, \Theta_t)$ such that $\theta_{jk}(t) = \theta_k(t) - \theta_j(t)$. We denote

$\theta_j(t) = \sum_{\alpha=1}^n \theta_j^\alpha(z_j,t)\frac{\partial}{\partial z_j^\alpha}$. It can then be checked that on $(U_j \cap U_k) \times I \neq \emptyset$ of M_I, $-\sum_{\alpha=1}^n \theta_j^\alpha(z_j,t)\frac{\partial}{\partial z_j^\alpha} + \left(\frac{\partial}{\partial t}\right)_j = -\sum_{\alpha=1}^n \theta_k^\alpha(z_k,t)\frac{\partial}{\partial z_k^\alpha} + \left(\frac{\partial}{\partial t}\right)_k$. Hence, we can define a vector field $v = -\sum_{\alpha=1}^n \theta_j^\alpha(z_j,t)\frac{\partial}{\partial z_j^\alpha} + \frac{\partial}{\partial t}$ globally on M_I. ■

REMARK $\theta_j(t)$ is *not* uniquely determined by $\{\theta_{jk}(t)\}$, and it may not be differentiable in t.

Theorem 2.3 *If dim $H^1(M_t, \Theta_t)$ is independent of $t \in I \subseteq \mathbb{R}^m, m \geq 1$, and $\theta(t) = 0$, then there exists $\{\theta_j(t)\}$ with $\delta\{\theta_j(t)\} = \{\theta_{jk}(t)\}$ such that each component $\theta_j^\alpha(t)$ is C^∞ in $z_j^1, ..., z_j^n, t$.*

The proof of this theorem is deferred to **chapter 6** where sufficient machinery is available. Assuming this theorem holds, we now have

Theorem 2.4 *Let $M_t = \varpi^{-1}(t), \quad t \in B \subseteq \mathbb{R}$. If dim $H^1(M_t, \Theta_t)$ is t-independent, and $\theta(t) = \frac{dM_t}{dt} = 0$ identically, then $(\mathcal{M}, B, \varpi)$ is locally trivial.*

PROOF OF THEOREM 2.4

Without loss of generality, assume $M = \varpi^{-1}(0)$. Let $p \in M$ and assume also that $p \in U_i \times 0$, with local coordinates $(\eta_i(p), 0)$. Then, choosing C^∞ $\theta_j(t)$ on $U_j \times I \subseteq M_I$ as assured by **Theorem 2.3**, the following system

$$\begin{cases} \dfrac{dz_j^\alpha}{dt} = -\theta_j^\alpha(z_j^1, ..., z_j^n, t), \qquad \alpha = 1, ..., n \\ z_i^\alpha(0) = \eta_i^\alpha(p), \qquad \alpha = 1, ..., n \end{cases} \tag{2.6}$$

has a unique solution $z_j^\alpha(t) = z_j^\alpha(p,t)$. This defines a C^∞ curve $\gamma_p : I \to M_I$ through p, where $\gamma_p(t) = (z_j(p,t), t), \forall t \in I$. This, in turns, by the uniqueness of the solution, defines a diffeomorphism $\Phi : M \times I \to M_I$, with $\Phi(p,t) = (z_j(p,t), t), \forall p \in M, \forall t \in I$.

What is remained to show is that the map $\Phi : M \times t \to M_t$ is biholomorphic.

Since Φ is onto M_t, one only needs to check if it is holomorphic in p. Now, by (2.6), and writing $z_j^\alpha = z_j^\alpha(\eta_i^1, ..., \eta_i^n, t)$, we have

$$\begin{aligned} \frac{d}{dt} z_j^\alpha(\eta_i, t) &= -\theta_j^\alpha(z_j(\eta_i, t), t) \\ \Rightarrow \frac{d}{dt}\frac{\partial z_j^\alpha(\eta_i, t)}{\partial \overline{\eta}_i^\lambda} &= -\sum_{\beta=1}^{n} \frac{\partial \theta_j^\alpha}{\partial z_j^\beta}(z_j(\eta_i, t), t))\frac{\partial z_j^\beta(\eta_i, t)}{\partial \overline{\eta}_i^\lambda} \end{aligned}$$

Hence, $w_j^\alpha(t) := \frac{\partial z_j^\alpha}{\partial \overline{\eta}_i^\alpha}(\eta_i, t)$ is a solution to the system

$$\begin{cases} \dfrac{d}{dt} w_j^\alpha(t) = -\displaystyle\sum_{\beta=1}^{n} \frac{\partial \theta_j^\alpha}{\partial z_j^\beta}(z_j(\eta_i, t), t) w_j^\beta(t), & \alpha = 1, ..., n \\ w_j^\alpha(0) = 0 \end{cases}$$

(**NOTE:** By definition, $w_j^\alpha(0) = \frac{\partial z_j^\alpha}{\partial \overline{\eta}_i^\lambda}(\eta_i, 0) = \frac{\partial \eta_j^\alpha}{\partial \overline{\eta}_i^\lambda} = 0$. Thus, the above initial condition is satisfied). Hence, by the uniqueness of the solution, we have $w_j^\alpha(t) \equiv 0 \Rightarrow \frac{\partial z_j^\alpha}{\partial \overline{\eta}_i^\lambda}(\eta_i, t) \equiv 0 \Rightarrow z_j^\alpha(p, t)$ is holomorphic in p. ■

Theorem 2.5 *If dim $H^1(M_t, \Theta_t)$ is independent of t and $\rho_t \equiv 0$, then $(\mathcal{M}, B, \varpi)$ is locally trivial.*

PROOF OF THEOREM 2.5

Without loss of generality, consider the case $\mathcal{M}_I := \varpi^{-1}(I) = \bigcup_{j=1}^{l} U_j \times I$ where I is a multi-interval in B. Decompose I into $I = I^{m-1} \times I_m$, where I_m is an interval, then $(\varpi^{-1}(I^{m-1}), I^{m-1}, \varpi)$ is a differentiable family. The statement can be proved by induction on m. The case when $m = 1$ is done as in **Theorem 2.4**. Suppose $\varpi^{-1}(I^{m-1})$ is equivalent to $M \times I^{m-1}$, then we can define a diffeomorphism $\Phi : M_I \to \varpi^{-1}(I^{m-1}) \times I_m$, biholomorphic for each fixed t, using the solutions of some system of ODEs as in the proof of **Theorem 2.4.** ■

2.3 Complex Analytic Family and Deformations

In this section, we consider the situation where the parameters are complex, with biholomorphic instead of diffeomorphic transition mappings.

Definition 2.4 *Given a domain $B \subseteq \mathbb{C}^m$, the family $\{M_t | t \in B\}$ of complex manifolds M_t is called a* ***complex analytic family of compact complex manifolds*** *if there is a complex manifold $\mathcal{M}$ and a holomorphic map $\varpi : \mathcal{M} \to B$ such that*

1. *The rank of the Jacobian of $\varpi = m$ at each point of $\mathcal{M}$.*

2. *$\varpi^{-1}(t) = M_t$ is a compact submanifold of $\mathcal{M}$ for each $t \in B$.*

Definition 2.5 *Suppose M and N are compact complex manifolds. Then N* ***is a deformation of*** *M if there exists a complex analytic family $(\mathcal{M}, B, \varpi)$ such that $M = \varpi^{-1}(t_0)$ and $N = \varpi^{-1}(t_1)$ for some $t_0, t_1 \in B$.*

Theorem 2.6 *Given any complex analytic family $(\mathcal{M}, B, \varpi)$, and any $t_0 \in B$, $M_t = \varpi^{-1}(t)$ is diffeomorphic to $M_{t_0} = \varpi^{-1}(t_0), \forall t \in B$.*

PROOF OF THEOREM 2.6

First consider the case where $m = 1$. Refer to the proof of **Lemma 2.1**. By using a partition of unity $\{\rho_j(x_j, t)\}$ subordinate to $\{\mathcal{U}_j\}$, one can construct a C^∞ vector field on $\mathcal{M}$ which is given on $\mathcal{U}_j$ by $\sum_{\alpha=1}^n \theta_j^\alpha(x_j, t)\frac{\partial}{\partial x_j^\alpha} + \frac{\partial}{\partial t}$ (by taking $\theta_j^\alpha(x_j, t) = \sum_{k \neq j} \rho_k(x_k, t)\frac{\partial}{\partial t} f_{jk}^\alpha(x_k, t)$, $x_k = f_{kj}(x_j, t)$). Then by considering the solutions to (2.6) one should obtain a diffeomorphism $\Phi : M_{t_0} \times I \to M_I$ for each multi-interval I in B. The global diffeomorphism $\mathcal{M} \cong M_{t_0} \times B$ is obtained by the connectedness of B. The general case $m \geq 1$ is done by induction as outlined in the proof of **Theorem 2.5**. ■

Again, let $z_j^\alpha = f_{jk}^\alpha(z_k, t), \alpha = 1, ..., n$ be a local coordinate system of $\mathcal{M}$, then for any $\frac{\partial}{\partial t} = \sum_{\lambda=1}^m c_\lambda \left(\frac{\partial}{\partial t_\lambda}\right) \in T_t(B)$, where $c_\lambda \in \mathbb{C}$, we can define $\theta(t) \in H^1(M_t, \Theta_t)$ by $\{\theta_{jk}(t)\}$, symbolically as in (2.4). This infinitesimal deformation $\theta(t)$ along $\frac{\partial}{\partial t}$ is denoted by $\frac{\partial M_t}{\partial t}$. The relation $\frac{\partial}{\partial t} \to \frac{\partial M_t}{\partial t}$ defines a $\mathbb{C}$-linear map $\tilde{\rho}_t : T_t(B) \to H^1(M_t, \Theta_t)$.

Note that f_{jk}^{α}'s are holomorphic in all variables. In particular, $\frac{\partial f_{jk}^{\alpha}}{\partial \bar{t}} = 0 \Rightarrow \tilde{\rho}_t\left(\frac{\partial}{\partial \bar{t}}\right) = 0$, and hence ρ_t of $(\mathcal{M}, B, \varpi)$ considered as a differentiable family coincides with $\tilde{\rho}_t$. Hence, from now on we write ρ_t in both situations. Local triviality is also similarly defined as in **Definition 2.3**, except that we require a biholomorphic(instead of diffeomorphic) equivalence between the families.

Theorem 2.7 *If dim $H^1(M_t, \Theta_t)$ is independent of t, $\rho_t \equiv 0$, then $(\mathcal{M}, B, \varpi)$ is locally trivial.*

The proof is a bit different from that of **Theorem 2.5**. Again we write $\mathcal{M}_\Delta = \varpi^{-1}(\Delta) = \bigcup_{j=1}^{l} U_j \times \Delta$, and decompose $\Delta = \Delta^{m-1} \times \Delta_m$, where Δ^{m-1} is a polydisc with polyradii $(r, ..., r)$. Define $\theta_{jk}(t)$ as in (2.3). From **Theorem 2.3**, we can assume there is a corresponding 0-cochain $\{\theta_j(t)\}$ where

$$\theta_j(t) = \sum_{\alpha=1}^{n} \theta_j^{\alpha}(z_j, t) \frac{\partial}{\partial z_j^{\alpha}}, \tag{2.7}$$

whose coefficients θ_j^{α} are C^∞ functions. We can then set up a system symbolically identical to that shown in (2.6), which gives a unique C^∞ solution and hence a C^∞ map $\Phi : \varpi^{-1}(\Delta^{m-1}) \times \Delta_m \to \mathcal{M}_\Delta$. If we can prove that θ_j^{α}'s can be chosen to be holomorphic in t, then Φ becomes a biholomorphism and we are done. To achieve this aim, we need a lemma which will be proved later:

Lemma 2.2 *Suppose $(\mathcal{M}, B, \varpi)$ is a complex analytic family with $d :=$ dim $H^0(M_t, \Theta_t)$ t-independent, then for a sufficiently small polydisc $\Delta \subseteq B$ containing 0, there exists a basis $\{\phi_1(t), ..., \phi_d(t)\}$ of $H^0(M_t, \Theta_t)$, where $\phi_q(t) = \sum_{\alpha=1}^{n} \phi_{qj}^{\alpha}(z_j, t) \frac{\partial}{\partial z_j^{\alpha}}$, and ϕ_{qj}^{α}'s are holomorphic in all variables.*

PROOF OF THEOREM 2.7

The proof that we can choose holomorphic θ_j^{α} can be divided into 3 main steps:

- $\eta_\lambda(t) := \sum_{\alpha=1}^{n} \frac{\partial \theta_j^{\alpha}}{\partial \bar{t}_\lambda}(z_j, t) \frac{\partial}{\partial z_j^{\alpha}}$ *on $U_j \times t \subseteq M_t$ is a well-defined C^∞ vector field* *on* M_t. Indeed, one can show that $\displaystyle\sum_{\beta=1}^{n}\frac{\partial\theta_k^\beta}{\partial\bar{t}_\lambda}(z_k,t)\frac{\partial}{\partial z_k^\beta}=\sum_{\alpha=1}^{n}\frac{\partial\theta_j^\alpha}{\partial\bar{t}_\lambda}(z_j,t)\frac{\partial}{\partial z_j^\alpha}$ by using the fact that $\theta_{jk}(t)=\theta_k(t)-\theta_j(t)$ and each $\theta_{jk}^\alpha(t)$ is holomorphic in all t_λ.

- $\theta_j^\alpha(z_j,t)$*'s are holomorphic in all variables if dim* $H^0(M,\Theta)=0$, *where* $M=\varpi^{-1}(0)$. We consider $(\mathcal{M},B,\varpi)$ as a differentiable family, then $\rho_t\equiv 0$, implying $(\mathcal{M},B,\varpi)$ is locally trivial as a differentiable family.
  $\Rightarrow \forall t\in B$, $M_t=\varpi^{-1}(t)$ is biholomorphic to M.
  $\Rightarrow$ dim $H^0(M_t,\Theta_t)=$ dim $H^0(M,\Theta)=0$ by the assumption. Since $\eta_\lambda(t)\in H^0(M,\Theta)$, we have $\eta_\lambda(t)\equiv 0\Rightarrow \frac{\partial\theta_j^\alpha}{\partial\bar{t}_\lambda}(z_j,t)\equiv 0$ and we are done.

- *If dim* $H^0(M,\theta)\geq 1$, *we can still choose holomorphic functions* $\tilde{\theta}_j^\alpha(z_j,t)$. We need to make use of **Lemma 2.2** here. From the argument above, we have $d=$ dim $H^0(M_t,\Theta_t)=$ dim $H^0(M,\Theta)\geq 1$. Thus, we can write $\eta_\lambda(t)=\sum_{q=1}^{d}c_{q\lambda}(t)\phi_q(t)$ for some $c_{q\lambda}(t)$ which are C^∞ in $t_1,...,t_m$.

$$\begin{aligned}\because 0 &= \overline{\partial}_t\overline{\partial}_t\theta_j^\alpha(z_j,t)=\overline{\partial}_t\Big(\sum_{\lambda=1}^{m}\frac{\partial\theta_j^\alpha}{\partial\bar{t}_\lambda}(z_j,t)d\bar{t}_\lambda\Big)\\ &= \overline{\partial}_t\Big(\sum_{\lambda=1}^{m}\sum_{q=1}^{d}c_{q\lambda}(t)\phi_{qj}^\alpha(z_j,t)d\bar{t}_\lambda\Big)=\sum_{q=1}^{d}\phi_{qj}^\alpha(z_j,t)\overline{\partial}_t\Big(\sum_{\lambda=1}^{m}c_{q\lambda}(t)d\bar{t}_\lambda\Big)\end{aligned}$$

$\Rightarrow \overline{\partial}_t\Big(\sum_{\lambda=1}^{m}c_{q\lambda}(t)d\bar{t}_\lambda\Big)=0$ on the sufficiently small $\Delta\subseteq B$ as in **Lemma 2.2**.
$\Rightarrow\ \exists c_q(t)\in C^\infty(\Delta)$ such that $\overline{\partial}_t c_q(t)=\sum_{\lambda=1}^{m}c_{q\lambda}(t)d\bar{t}_\lambda$, by Dolbeault's lemma.

Now, define $\tilde{\theta}_j(t)=\theta_j(t)-\sum_{q=1}^{d}c_q(t)\phi_q^\alpha(t)$, then it is easy to check that $\overline{\partial}_t\tilde{\theta}_j^\alpha=0$; and $\tilde{\theta}_k(t)-\tilde{\theta}_j(t)=\theta_k(t)-\theta_j(t)=\theta_{jk}(t)$. ■

REMARK

1. **Change of Parameters** Given a complex analytic family $(\mathcal{M},B,\varpi)$ and a holomorphic map $h:D\to B, h(s)=t, B\subseteq\mathbb{C}^n$ is a domain, we can define another $(\mathcal{N},D,\pi)$, called the complex analytic family induced

from $(\mathcal{M}, B, \varpi)$ by h. Essentially, this is a family consisting of complex manifolds $M_{h(s)} \in \mathcal{M}$, where $s \in D$ with $\pi(M_{h(s)}) = s$. Thus $\forall \frac{\partial}{\partial s} \in T_s(D)$, we have $\dfrac{\partial M_{h(s)}}{\partial s} = \displaystyle\sum_{\alpha=1}^{m} \frac{\partial t_\lambda}{\partial s}\frac{\partial M_t}{\partial t}$.

2. This condition that $\dim H^1(M_t, \Theta_t)$ is t-independent cannot be dropped when we try to derive local triviality from a zero infinitesimal deformation. The family of Hopf Surfaces of dimensions 2 will illustrate this point. Fix $\alpha \in \mathbb{C}$ where $0 < |\alpha| < 1$. For each $t \in \mathbb{C}$, let

$$M_t = (\mathbb{C}^2 - \{0\})/\{g_t^m : \mathbb{C}^2 - \{0\} \to \mathbb{C}^2 - \{0\} | m \in \mathbb{Z}, g_t(z_1, z_2) = (\alpha z_1 + t z_2, \alpha z_2)\},$$

which is a complex manifold of dimension 2. The collection $\mathcal{M} = \{M_t | t \in \mathbb{C}\}$ form a complex analytic family $(\mathcal{M}, \mathbb{C}, \varpi)$. In this family, all $M_t, t \neq 0$ are biholomorphic equivalent, while M_0 has a complex structure different from others.

The family $(\mathcal{N}, \mathbb{C}, \pi)$ induced from $(\mathcal{M}, \mathbb{C}, \varpi)$ by $s \to t = s^2$ has the infinitesimal deformation

$$\rho_s\Big(\frac{d}{ds}\Big) = \frac{dM_{s^2}}{ds} = \frac{dt}{ds}\frac{dM_t}{dt} = 2s\frac{dM_t}{dt}$$

Thus, $\rho_0 = 0$; For all $s \neq 0$ (i.e. $t = s^2 \neq 0$), we have $\frac{dM_t}{dt} = 0$ since $(\mathcal{M}, \mathbb{C}, \varpi)$ is locally trivial at such t. In other words, $\rho_s = 0, \forall s \in \mathbb{C}$.

However, $(\mathcal{N}, \mathbb{C}, \pi)$ is not locally trivial, since $\pi^{-1}(0) = M_0$ is not biholomorphically equivalent to $\pi^{-1}(s) = M_t$, where $t = s^2 \neq 0$. In fact, dim $H^1(M_t, \Theta_t)$ is t-dependent, having a value of 4 when $t = 0$, and 2 otherwise (This has been fully worked out in [8], p.427-36).

Chapter 3

Existence Theorem

In this chapter, we are to consider the following problem:

Given $\theta \in H^1(M, \Theta)$, where M is a compact complex manifold, does there exist a complex analytic family $(\mathcal{M}, B, \varpi), 0 \in B \subseteq \mathbb{C}$ such that

$$\varpi^{-1}(0) = M, \qquad \left(\frac{dM_t}{dt}\right)_{t=0} = \theta?$$

3.1 Obstructions as a Necessary Condition

Suppose such a family exists, then define the infinitesimal deformation $\theta(t) = \frac{dM_t}{dt} \in H^1(M_t, \Theta_t)$ by $\{\theta_{jk}(t)\} \in Z^1(\mathfrak{U}_t, \Theta_t)$, ($\mathfrak{U}_t$ is an open cover of polydiscs on M_t, c.f. §2.1)

$$\theta_{jk}(t) = \sum_{\alpha=1}^{n} \theta_{jk}^{\alpha}(z_j, t) \frac{\partial}{\partial z_j^{\alpha}}, \qquad \theta_{jk}^{\alpha}(z_j, t) = \frac{\partial f_{jk}^{\alpha}(z_k, t)}{\partial t}, z_k = f_{kj}(z_j, t)$$

Let $\dot{\theta}_{ik}^{\alpha}(z_i, t) = \dfrac{\partial \theta_{ik}^{\alpha}(z_i, t)}{\partial t}$. Differentiate the following relation with respect to t,

$$f_{ik}^{\alpha}(z_k, t) = f_{ij}^{\alpha}(f_{jk}(z_k, t), t), \qquad \alpha = 1, ..., n \tag{3.1}$$

$$\Rightarrow \theta_{ik}^{\alpha}(z_i, t) = \theta_{ij}^{\alpha}(z_i, t) + \sum_{\beta=1}^{n} \frac{\partial f_{ij}^{\alpha}(z_j, t)}{\partial z_j^{\beta}} \theta_{jk}^{\beta}(z_j, t) \quad (\because z_j = f_{jk}(z_k, t))$$

Using the fact that $\left(\frac{\partial}{\partial t}\right)_j = \sum_{\beta=1}^n \left(\frac{\partial z_i^\beta}{\partial t}\right)_j \frac{\partial}{\partial z_i^\beta} + \left(\frac{\partial}{\partial t}\right)_i \left(\frac{\partial z_i^\beta}{\partial t}\right)_j = \frac{\partial f_{ij}^\beta(z_j,t)}{\partial t}$, and differentiate once more with respect to t, we have

$$[\theta_{ij}(t)\cdot\theta_{ik}^\alpha(z_i,t)+\dot\theta_{ik}^\alpha(z_i,t)]$$
$$= [\theta_{ij}(t)\cdot\theta_{ij}^\alpha(z_i,t)+\dot\theta_{ij}^\alpha(z_i,t)] + [\theta_{jk}(t)\cdot\theta_{ij}^\alpha(z_i,t)+\sum_{\beta=1}^n \frac{\partial z_i^\alpha}{\partial z_j^\beta}\dot\theta_{jk}^\beta(z_j,t)]$$
$$\Rightarrow \theta_{ij}(t)\cdot[\theta_{ik}^\alpha(z_i,t)-\theta_{ij}^\alpha(z_i,t)] - \theta_{jk}(t)\cdot\theta_{ij}^\alpha(z_i,t) = \dot\theta_{ij}^\alpha(z_i,t)-\dot\theta_{ik}^\alpha(z_i,t)+\sum_{\beta=1}^n \frac{\partial z_i^\alpha}{\partial z_j^\beta}\dot\theta_{jk}^\beta(z_j,t)$$
$$\Rightarrow \theta_{ij}(t)\cdot\sum_{\beta=1}^n \frac{\partial z_i^\alpha}{\partial z_j^\beta}\theta_{jk}^\beta(z_j,t) - \theta_{jk}(t)\cdot\theta_{ij}^\alpha(z_i,t) = \dot\theta_{ij}^\alpha(z_i,t)-\dot\theta_{ik}^\alpha(z_i,t)+\sum_{\beta=1}^n \frac{\partial z_i^\alpha}{\partial z_j^\beta}\dot\theta_{jk}^\beta(z_j,t)$$
$$\Rightarrow [\theta_{ij}(t),\theta_{jk}(t)] = \dot\theta_{ij}(t)-\dot\theta_{ik}(t)+\dot\theta_{jk}(t),$$

where $[v,u] = \sum_\alpha (v\cdot u_j^\alpha - u\cdot v_j^\alpha)\frac{\partial}{\partial z_j^\alpha}$. In particular, $[\theta_{ij}(0),\theta_{jk}(0)] = \dot\theta_{ij}(0) - \dot\theta_{ik}(0)+\dot\theta_{jk}(0)$. In the cohomology class level, $\{[\theta_{ij}(0),\theta_{jk}(0)]\}\in Z^2(\mathfrak{U},\Theta)$ and $\{\dot\theta_{jk}(0)\}\in C^1(\mathfrak{U},\Theta) \Rightarrow [\theta(0),\theta(0)] = 0$ in $H^2(\mathfrak{U},\Theta)$.

Theorem 3.1 *If the existence problem is solved, then it is necessary to have $[\theta,\theta]=0$. $[\theta,\theta]\in H^2(M,\Theta)$ is called the (primary) obstruction to deformation of M.*

REMARK More obstructions on θ can be obtained by differentiating (3.1) with respect to t by arbitrarily many times.

3.2 The Existence Theorem

We state the existence theorem established by Kodaira and Spencer [6],[8]:

Theorem 3.2 *Let M be a compact complex manifold and suppose $H^2(M,\Theta)=0$, then there exists a complex analytic family $(\mathcal{M},B,\varpi)$ with $0\in B\subseteq\mathbb{C}^m$ such that*

1. $\varpi^{-1}(0) = M$;

2. $\rho_0 : \frac{\partial}{\partial t} \to \left(\frac{\partial M_t}{\partial t}\right)$ *(with* $M_t = \varpi^{-1}(t)$*) is an isomorphism of* $T_0(B)$ *onto* $H^1(M, \Theta)$

One way to prove this theorem is the determination of a convergent power series based on the given data, and the use of a theorem by Newlander and Nirenberg [12]. Note that the hypothesis that $H^2(M, \Theta) = 0$ automatically leads to zero obstruction.

Suppose **Theorem 3.2** is solved for some $(\mathcal{M}, B, \varpi)$, then take a sufficiently small polydisc $\Delta \subseteq B$ such that $\varpi^{-1}(\Delta) = \mathcal{M}_\Delta = \bigcup_j U_j \times \Delta$. Denote a point in U_j by $\zeta_j = (\zeta_j^1, ..., \zeta_j^m)$ and let $\zeta_j^\alpha = f_{jk}^\alpha(\zeta_k, t)$ be the transition mapping. Since there exists a diffeomorphism $\Psi : M \times \Delta \to \mathcal{M}_\Delta$ (by **Theorem 1.6**), we have

$$\varpi \circ \Psi(z, t) = t, \qquad \forall z \in M, \forall t \in \Delta$$

Thus, we can identify $\mathcal{M}_\Delta$ with $M \times \Delta$ via Ψ. Under such identification, $\mathcal{M}_\Delta$ is a complex manifold whose local coordinates are $(\zeta_j^1(z, t), ..., \zeta_j^n(z, t)), j \in \mathbb{N}$. Moreover, we have $M = M_0$, whose local complex coordinates are $(z^1, ..., z^n)$.

Back to the proof of the existence theorem. Given a basis $\{\beta_1, ..., \beta_m\}$ of $H^1(M, \Theta)$, one need to find a family $\{\phi(t) \mid t \in \Delta\}$ on some small $\Delta \subseteq B$ such that

$$(*)\begin{cases} \overline{\partial}\phi(t) = \frac{1}{2}[\phi(t), \phi(t)] \\ \phi(0) = 0, \qquad \left(\frac{\partial \phi}{\partial t_\lambda}\right)_{\lambda=1} = \beta_\lambda, \quad \lambda = 1, ..., m \end{cases}$$

where for any (vector-valued) (0,1)-forms $\phi = \sum_\lambda \phi^\lambda \frac{\partial}{\partial z^\lambda} = \sum_{\lambda,\nu} \phi_{\overline{\nu}}^\lambda(z, t) d\overline{z}^\nu \frac{\partial}{\partial z^\lambda}$ and $\psi = \sum_\lambda \psi^\lambda \frac{\partial}{\partial z^\lambda} = \sum_{\lambda,\nu} \psi_{\overline{\nu}}^\lambda(z, t) d\overline{z}^\nu \frac{\partial}{\partial z^\lambda}$, the Lie bracket is defined by

$$[\phi, \psi] = \sum_{\lambda,\mu} (\phi^\mu \wedge \frac{\partial \psi^\lambda}{\partial z^\mu} + \psi^\mu \wedge \frac{\partial \phi^\lambda}{\partial z^\mu}) \frac{\partial}{\partial z^\lambda}, \qquad \frac{\partial \psi^\lambda}{\partial z^\mu} = \sum_\nu \frac{\partial \psi_{\overline{\nu}}^\lambda}{\partial z^\mu}(z) d\overline{z}^\nu$$

Suppose we have got such $\phi(t)$, which is treated as a (0,1)-form on $M \times B$ by writing

$$\phi(t) = \sum_\lambda \left(\sum_\nu \phi_{\overline{\nu}}^\lambda(z, t) d\overline{z}^\nu + \sum_\mu 0 d\overline{t}_\mu \right) \frac{\partial}{\partial z^\lambda} + \sum_\mu 0 \frac{\partial}{\partial t_\mu}$$

(We shall construct this $\phi(t)$ in §3.3). Let $L_\nu = \frac{\partial}{\partial \overline{z}^\nu} - \sum_\lambda \phi^\lambda_{\overline{\nu}} \frac{\partial}{\partial z^\lambda}$, $\nu = 1, ..., n$, then we have

Lemma 3.1 $L_1, ..., L_n, \overline{L}_1, ..., \overline{L}_n, \frac{\partial}{\partial t_1}, ..., \frac{\partial}{\partial t_m}, \frac{\partial}{\partial \overline{t}_1}, ..., \frac{\partial}{\partial \overline{t}_m}$ *are linearly independent for some sufficiently small* $\Delta \subseteq B$.

PROOF OF LEMMA 3.1

Since by hypothesis, $\phi(0) = 0$, we have $\det(\delta^\lambda_\nu - \sum_\mu \phi^\mu_{\overline{\nu}} \phi^\lambda_{\overline{\mu}}) \neq 0$ when $\Delta \subseteq B$ is small. Thus, $L_i, \overline{L}_i$, $i = 1, ..., n$ are linearly independent, which suffices to conclude the proof. ■

From now on, we take $\Delta \subseteq B$ small enough so that **Lemma 3.1** holds.

Lemma 3.2 $L_\tau L_\nu - L_\nu L_\tau = 0, \forall \tau, \nu = 1, ..., n$.

PROOF OF LEMMA 3.2

$$
\begin{aligned}
\because \quad & \overline{\partial}\phi(t) = \frac{1}{2}[\phi(t), \phi(t)] \\
\Leftrightarrow \quad & \overline{\partial}\Big(\sum_{\lambda,\nu} \phi^\lambda_{\overline{\nu}} d\overline{z}^\nu \frac{\partial}{\partial z^\lambda}\Big) = \sum_{\lambda,\mu} \Big(\phi^\mu \wedge \frac{\partial \phi^\lambda}{\partial z^\mu}\Big) \frac{\partial}{\partial z^\lambda} \\
\Leftrightarrow \quad & \sum_{\lambda,\nu,\tau} \Big(\frac{\partial \phi^\lambda_{\overline{\nu}}}{\partial \overline{z}^\tau} d\overline{z}^\tau \wedge d\overline{z}^\nu\Big) \frac{\partial}{\partial z^\lambda} = \sum_{\lambda,\mu} \Big(\big(\sum_\tau \phi^\mu_{\overline{\tau}} d\overline{z}^\tau\big) \wedge \big(\sum_\nu \frac{\partial \phi^\lambda_{\overline{\nu}}}{\partial z^\mu} d\overline{z}^\nu\big)\Big) \frac{\partial}{\partial z^\lambda} \\
\Leftrightarrow \quad & \frac{\partial \phi^\lambda_{\overline{\nu}}}{\partial \overline{z}^\tau} - \frac{\partial \phi^\lambda_{\overline{\tau}}}{\partial \overline{z}^\nu} = \sum_\mu \Big(\phi^\mu_{\overline{\tau}} \frac{\partial \phi^\lambda_{\overline{\nu}}}{\partial z^\mu} - \phi^\mu_{\overline{\nu}} \frac{\partial \phi^\lambda_{\overline{\tau}}}{\partial z^\mu}\Big)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
L_\tau L_\nu &= \Big(\frac{\partial}{\partial \overline{z}^\tau} - \sum_\lambda \phi^\lambda_{\overline{\tau}} \frac{\partial}{\partial z^\lambda}\Big)\Big(\frac{\partial}{\partial \overline{z}^\nu} - \sum_\mu \phi^\mu_{\overline{\nu}} \frac{\partial}{\partial z^\mu}\Big) \\
&= \frac{\partial^2}{\partial \overline{z}^\tau \partial \overline{z}^\nu} - \sum_\lambda \phi^\lambda_{\overline{\tau}} \frac{\partial^2}{\partial z^\lambda \partial \overline{z}^\nu} - \Big[\sum_\mu \phi^\mu_{\overline{\nu}} \frac{\partial^2}{\partial \overline{z}^\tau \partial z^\mu} + \frac{\partial \phi^\mu_{\overline{\nu}}}{\partial \overline{z}^\tau} \frac{\partial}{\partial z^\mu}\Big] \\
&\qquad + \sum_{\lambda,\mu} \Big(\phi^\lambda_{\overline{\tau}} \frac{\partial \phi^\mu_{\overline{\nu}}}{\partial z^\lambda} \frac{\partial}{\partial z^\mu} + \phi^\lambda_{\overline{\tau}} \phi^\mu_{\overline{\nu}} \frac{\partial^2}{\partial z^\lambda \partial z^\mu}\Big)
\end{aligned}
$$

$\therefore$ By exchanging the roles of τ and ν (as well as the indices μ and λ), we have

$$\begin{aligned} L_\tau L_\nu - L_\nu L_\tau &= -\sum_\mu \frac{\partial \phi^\mu_{\overline{\nu}}}{\partial \overline{z}^\tau}\frac{\partial}{\partial z^\mu} + \sum_\lambda \frac{\partial \phi^\lambda_{\overline{\tau}}}{\partial \overline{z}^\nu}\frac{\partial}{\partial z^\lambda} + \sum_{\mu,\lambda} \phi^\lambda_{\overline{\tau}}\frac{\partial \phi^\mu_{\overline{\nu}}}{\partial z^\lambda}\frac{\partial}{\partial z^\mu} - \sum_{\mu,\lambda} \phi^\mu_{\overline{\nu}}\frac{\partial \phi^\lambda_{\overline{\tau}}}{\partial z^\mu}\frac{\partial}{\partial z^\lambda} \\ &= \sum_\lambda \Big[-\frac{\partial \phi^\lambda_{\overline{\nu}}}{\partial \overline{z}^\tau} + \frac{\partial \phi^\lambda_{\overline{\tau}}}{\partial \overline{z}^\nu} + \sum_\mu \Big(\phi^\mu_{\overline{\tau}}\frac{\partial \phi^\lambda_{\overline{\nu}}}{\partial z^\mu} - \phi^\mu_{\overline{\nu}}\frac{\partial \phi^\lambda_{\overline{\tau}}}{\partial z^\mu}\Big)\Big]\frac{\partial}{\partial z^\lambda} = 0 \qquad \blacksquare \end{aligned}$$

We now state the Newlander-Nirenberg theorem which guarantees the existence of a complex structure:

Theorem 3.3 *If the partial differential operators $D_i, \overline{D}_i, 1 \le i \le n+m$ are linearly independent and satisfy $D_\tau D_\nu - D_\nu D_\tau = 0$, then the system of equations*

$$D_\nu f = 0, \nu = 1, ..., n \tag{3.2}$$

$$D_{n+\mu} f = 0, \mu = 1, ..., m \tag{3.3}$$

has $n+m$ linearly independent C^∞ solutions $f = \zeta^\alpha = \zeta^\alpha_j(z,t)$ on $U_j \times \Delta_\varepsilon$ (Possibly $\Delta_\varepsilon \subsetneqq \Delta$), i.e. $\det \frac{\partial(\zeta^1,...,\zeta^{n+m},\overline{\zeta}^1,...,\overline{\zeta}^{m+n})}{\partial(z^1,...,z^{n+m},\overline{z}^1,...,\overline{z}^{m+n})} \neq 0$.

It is clear that by **Lemma 3.1** and **3.2**, the hypothesis of the Newlander-Nirenberg theorem is satisfied (by taking $D_\nu = L_\nu,\ D_{n+\mu} = \frac{\partial}{\partial \overline{t}_\mu}$). $D_\tau D_\nu - D_\nu D_\tau = 0$ (or equivalently $\overline{\partial}\phi(t) = \frac{1}{2}[\phi(t), \phi(t)]$) is called the integrability condition for $D_\nu f = 0$.

Lemma 3.3 *A local C^∞ function f on M is holomorphic with respect to $\zeta^\alpha_j, 1 \le \alpha \le n$ if and only if $(\overline{\partial} - \phi(t))f = 0$, where $\Delta \subseteq B$ is sufficiently small.*

PROOF OF LEMMA 3.3

We first prove an auxillary result:

$$\begin{aligned} L_\nu \zeta^\alpha_j = 0 &\Leftrightarrow \frac{\partial \zeta^\alpha_j}{\partial \overline{z}^\nu} = \sum_\mu \phi^\mu_{\overline{\nu}} \frac{\partial \zeta^\alpha_j}{\partial z^\mu} \\ &\Leftrightarrow \sum_\nu \frac{\partial \zeta^\alpha_j}{\partial \overline{z}^\nu} d\overline{z}^\nu = \sum_{\nu,\mu} \phi^\mu_{\overline{\nu}} \frac{\partial \zeta^\alpha_j}{\partial z^\mu} d\overline{z}^\nu = \sum_\mu \phi^\mu \frac{\partial \zeta^\alpha_j}{\partial z^\mu} \end{aligned}$$

Write $f = f(\zeta_j^1, ..., \zeta_j^n)$ where $\zeta_j^\alpha = \zeta_j^\alpha(z,t)$, then

$$
\begin{aligned}
&(\overline{\partial} - \phi(t))f \\
&= \Big(\sum_{\nu} d\overline{z}^\nu \frac{\partial}{\partial \overline{z}^\nu} - \sum_{\mu} \phi^\mu \frac{\partial}{\partial z^\mu}\Big) f(\zeta_j^1(z,t), ..., \zeta_j^n(z,t)) \\
&= \sum_{\nu,\mu} d\overline{z}^\nu \Big(\frac{\partial f}{\partial \zeta_j^\alpha}\frac{\partial \zeta_j^\alpha}{\partial \overline{z}^\nu} + \frac{\partial f}{\partial \overline{\zeta}_j^\alpha}\frac{\partial \overline{\zeta}_j^\alpha}{\partial \overline{z}^\nu}\Big) - \sum_{\mu,\alpha} \phi^\mu \Big(\frac{\partial f}{\partial \zeta_j^\alpha}\frac{\partial \zeta_j^\alpha}{\partial z^\mu} + \frac{\partial f}{\partial \overline{\zeta}_j^\alpha}\frac{\partial \overline{\zeta}_j^\alpha}{\partial z^\mu}\Big) \\
&= \sum_{\alpha} \Big(\sum_{\nu} d\overline{z}^\nu \frac{\partial f}{\partial \overline{\zeta}_j^\alpha}\frac{\partial \overline{\zeta}_j^\alpha}{\partial \overline{z}^\nu} - \sum_{\mu} \phi^\mu \frac{\partial f}{\partial \overline{\zeta}_j^\alpha}\frac{\partial \overline{\zeta}_j^\alpha}{\partial z^\mu}\Big) \quad (\because \text{ By the auxillary result}) \\
&= \sum_{\alpha} \Big(\sum_{\nu} d\overline{z}^\nu \frac{\partial f}{\partial \overline{\zeta}_j^\alpha}\frac{\partial \overline{\zeta}_j^\alpha}{\partial \overline{z}^\nu} - \sum_{\mu,\nu} \phi_{\overline{\nu}}^\mu \frac{\partial f}{\partial \overline{\zeta}_j^\alpha}\frac{\partial \overline{\zeta}_j^\alpha}{\partial z^\mu} d\overline{z}^\nu\Big) \\
&= \sum_{\alpha,\nu} \Big(\overline{\frac{\partial \zeta_j^\alpha}{\partial z^\nu}} - \sum_{\mu} \phi_{\overline{\nu}}^\mu \overline{\frac{\partial \zeta_j^\alpha}{\partial \overline{z}^\mu}}\Big) d\overline{z}^\nu \frac{\partial f}{\partial \overline{\zeta}_j^\alpha} \\
&= \sum_{\alpha,\nu} \Big(\overline{\frac{\partial \zeta_j^\alpha}{\partial z^\nu}} - \sum_{\mu} \phi_{\overline{\nu}}^\mu \overline{\sum_{\lambda} \phi_{\overline{\mu}}^\lambda \frac{\partial \zeta_j^\alpha}{\partial z^\lambda}}\Big) d\overline{z}^\nu \frac{\partial f}{\partial \overline{\zeta}_j^\alpha} \quad (\because \text{ By the auxillary result}) \\
&= \sum_{\alpha,\nu,\lambda} (\delta_\nu^\lambda - \sum_{\mu} \phi_{\overline{\nu}}^\mu \overline{\phi_{\overline{\mu}}^\lambda}) \overline{\frac{\partial \zeta_j^\alpha}{\partial z^\lambda}} d\overline{z}^\nu \frac{\partial f}{\partial \overline{\zeta}_j^\alpha}
\end{aligned}
$$

$\therefore$ When Δ is sufficiently small, $\det\left(\frac{\partial \zeta_j^\alpha}{\partial z^\lambda}\right) \neq 0, \forall t \in \Delta$ (from **Lemma 3.1**) and we can assume $\det(\delta_\nu^\lambda - \sum \phi_{\overline{\nu}}^\mu \overline{\phi_{\overline{\mu}}^\lambda}) \neq 0$. Hence, we are done. ■

Therefore, together with the fact that ζ_j^α are linearly independent, ζ_j^α give local complex coordinates on $U_j \times \Delta_\varepsilon$. Now take $\zeta_j^{n+\mu}(z,t) = t_\mu, \mu = 1, ..., m$, which is a feasible solution to (3.3) when $D_{n+\mu} = \frac{\partial}{\partial t_\mu}$. Then, we can construct a holomorphic mapping $\varpi : \mathcal{M} \to \Delta_\varepsilon$ by defining $\varpi(\zeta_j^1(z,t), ..., \zeta_j^n(z,t), t) = t$. For each fixed t, $\varpi^{-1}(t)$ is a complex manifold whose local coordinates are $(\zeta_j^1(z,t), ..., \zeta_j^n(z,t))$. The linearly independence of $\zeta_j^\alpha(z,t)$ implies that $\varpi^{-1}(t) = M_{\phi(t)}$. Hence, we can conclude that $\mathcal{M} = \{M_{\phi(t)}\}$ forms a complex analytic family satisfying the requirements of the existence theorem.

REMARK

1. The idea of treating all M_t as the same differentiable manifold but with

different complex structures is crucial in simplifying the proof, and make a direct application of Newlander-Nirenberg theorem possible. Kodaira originally tried to prove the existence simply from the transition relation $f_{ik}(z_k, t) = f_{ij}(f_{jk}(z_k, t), t)$, but the results derived are complicated and does not lead to an affirmative conclusion [6].

2. The following converse is also true: Given a differentiable family $\mathcal{M}$, one can obtain the (0,1)-form $\phi(t)$ from the auxillary result: $\frac{\partial \zeta_j^\alpha}{\partial \bar{z}^\nu} = \sum_\mu \phi_{\bar{\nu}}^\mu \frac{\partial \zeta_j^\alpha}{\partial z^\mu}$, which reflects the complex structure of the family (**Lemma 3.3**) and whose derivatives with respect to the parameters give (the negative values of) the infinitesimal deformations.

3.3 Convergence Proof

Now we have only one problem left before we can prove the existence problem: How can we construct $\phi(t)$ so that (3.2) and (3.3) hold?

Let $\phi(t) = \sum \phi_{v_1 \dots v_m} t_1^{v_1} \dots t_m^{v_m}$. for any power series $P(t)$ in $t_1, \dots, t_m$, denote

$$P_{[v]}(t) = \sum_{v_1+\dots+v_m=v} P_{v_1\dots v_m} t_1^{v_1} \dots t_m^{v_m}$$

$$P^{[v]}(t) = P_{[0]}(t) + \dots + P_{[v]}(t) = \sum_{v_1+\dots+v_m \le v} P_{v_1\dots v_m} t_1^{v_1} \dots t_m^{v_m}$$

and write $P(t) \equiv_v Q(t)$ to denote any $P(t), Q(t)$ with $P^{[v]}(t) = Q^{[v]}(t)$.

Introduce a hermitian metric on the compact complex manifold M. This in turn defines an inner product $(\cdot, \cdot)$ on $\mathscr{L}^{0,q}(T)$, the set of all C^∞ $(0, q)$-forms on $T = T(M)$. Define ϑ to be the adjoint of $\overline{\partial}$ with respect to this $(\cdot, \cdot)$, and write $\square = \vartheta\overline{\partial} + \overline{\partial}\vartheta$. Denote the set of harmonic $(0, q)$-forms by $\mathbf{H}^{0,q}(T)$. In what follows, we will only be interested in the case where $q = 2$.

Lemma 3.4 *If $H^2(M,\Theta) = 0$, then (a) the Green operator $G = \Box^{-1}$ is well defined on $\mathscr{L}^{0,2}(T)$; and (b) $\forall \psi \in \mathscr{Z}^{0,2}_{\overline{\partial}}(T)$, i.e. a $\overline{\partial}$-closed $(0,2)$-form, $\psi = \overline{\partial}\vartheta G\psi$.*

Proof of Lemma 3.4

1. Since $H^2(M,\Theta) \cong \mathscr{Z}^{0,2}_{\overline{\partial}}(T)/\overline{\partial}\mathscr{L}^{0,1}(T) \cong \mathbf{H}^{0,2}(T)$, $H^2(M,\Theta) = 0$ implies that $\mathbf{H}^{0,2}(T) = 0$. Hence, if $\Box\psi = 0$, then $\psi = 0$. As $\mathscr{L}^{0,2}(T) = \mathbf{H}^{0,2}(T) \oplus \Box\mathscr{L}^{0,2}(T)$ in general, we have $\mathscr{L}^{0,2}(T) = \Box\mathscr{L}^{0,2}(T)$.

2. From (a), we have $\psi = \Box G\psi = (\overline{\partial}\vartheta + \vartheta\overline{\partial})G\psi \Rightarrow$ By hypothesis, $0 = \overline{\partial}\psi = \overline{\partial}\vartheta\overline{\partial}G\psi \Rightarrow ||\vartheta\overline{\partial}G\psi||^2 = (\vartheta\overline{\partial}G\psi, \vartheta\overline{\partial}G\psi) = (\overline{\partial}G\psi, \overline{\partial}\vartheta\overline{\partial}G\psi) = 0 \Rightarrow \psi = \overline{\partial}\vartheta G\psi$. ■

Lemma 3.5 *If $H^2(M,\Theta) = 0$, then $\phi_{v_1\ldots v_m} \in \mathscr{L}^{0,1}(T)$ can be found so that the formal series of $\phi(t)$ solves (*).*

Proof of Lemma 3.5

We consider $\phi(t) = \sum_{v\geq 0}\phi_{[v]}(t)$. Since the conditions $\phi(0) = 0$ and $\frac{\partial\phi}{\partial t_\lambda} = \beta_\lambda$ are to be satisfied, we can let $\phi_{[0]}(t) = 0$ and $\phi_{[1]}(t) = \sum_\lambda \beta_\lambda t_\lambda$. For $v \geq 2$, since $[\phi_{[u]}(t), \phi_{[v]}(t)]$ is homogeneous and of degree $u+v$, it suffices to solve

$$\overline{\partial}\phi^{[v]}(t) \equiv_v \frac{1}{2}[\phi^{[v-1]}(t), \phi^{[v-1]}(t)], \qquad v = 2,3,... \tag{3.4}$$

We proceed by induction on v. Suppose (3.4) is true for some $v \geq 2$, then let $\psi_{[v+1]}(t)$ be the $(v+1)$-th *homogeneous* part of $\frac{1}{2}[\phi^{[v]}(t), \phi^{[v]}(t)] - \overline{\partial}\phi^{[v]}(t)$, then

$$\begin{aligned}\overline{\partial}\psi_{[v+1]}(t) &\equiv_{v+1} \overline{\partial}\big(\frac{1}{2}[\phi^{[v]}(t), \phi^{[v]}(t)] - \overline{\partial}\phi^{[v]}(t)\big) = \frac{1}{2}\overline{\partial}[\phi^{[v]}(t), \phi^{[v]}(t)] = [\overline{\partial}\phi^{[v]}(t), \phi^{[v]}(t)] \\ &\equiv_{v+1} \frac{1}{2}[[\phi^{[v]}(t), \phi^{[v]}(t)], \phi^{[v]}(t)] \qquad (\because \overline{\partial}\phi^{[v]}(t) \equiv_v \frac{1}{2}[\phi^{[v]}(t), \phi^{[v]}(t)]) \\ &= 0\end{aligned}$$

Since $H^2(M,\Theta)=0$, we have, by **Lemma 3.4**, $\psi_{[v+1]}(t)=\overline{\partial}\vartheta G\psi_{[v+1]}(t)$. Define $\phi_{[v+1]}(t)=\vartheta G\psi_{[v+1]}(t)$, then $\overline{\partial}\phi_{[v+1]}(t)=\psi_{[v+1]}(t)$ and

$$\begin{aligned}\overline{\partial}\phi^{[v+1]}(t) &= \overline{\partial}\phi^{[v]}(t)+\overline{\partial}\phi_{[v+1]}(t)=\overline{\partial}\phi^{[v]}(t)+\psi_{[v+1]}(t)\\ &\equiv_{v+1}\frac{1}{2}[\phi^{[v]}(t),\phi^{[v]}(t)],\end{aligned}$$

by the definition of $\psi_{[v+1]}(t)$. ∎

Finally we need to prove that the formal series does converge. We make use of the Hölder norms to be defined below:

Definition 3.1 *1. Given any domain $U\subseteq\mathbb{C}^n\cong\mathbb{R}^{2n}$. and C^∞ $f=f(x^1,...,x^{2n}):U\to\mathbb{C}$,*

$$|f|^U_{k+\alpha}:=\sum_{h=0}^{k}\sum_{D^h}\sup_{x\in U}|D^hf(x)|+\sum_{D^k}\sup_{x,y\in U}\frac{|D^kf(x)-D^kf(y)|}{|x-y|^\alpha},\ k\geq 0,\alpha\in(0,1),$$

where $|x-y|=\sqrt{\sum_v|x^v-y^v|^2}$; $\sum_{D^h}$ is the summation over all partial differential operator D^h of order h.

2. Suppose $M=\cup U_j$, where $U_j=\{z_j\mid |z_j^1|,...,|z_j^n|<1\}$, then $\forall\phi\in\mathscr{L}^{0,q}(T)$, represent ϕ on U_j in the form

$$\phi=\sum_\lambda\frac{1}{q!}\sum\phi^\lambda_{j\overline{v_1}...\overline{v_q}}(x_j)d\overline{z_j}^{v_1}\wedge...\wedge d\overline{z_j}^{v_q}\frac{\partial}{\partial z_j^\lambda},\quad z_j^v=x_j^{2v-1}+ix_j^{2v}$$

Under such representation, define

$$|\phi|_{k+\alpha}=\max_j\max_{\lambda,v_1,...,v_q}|\phi^\lambda_{j\overline{v_1}...\overline{v_q}}|^{U_j}_{k+\alpha};\qquad |\phi|_0=\max_j\max_{\lambda,v_1,...,v_q}\sup_{x_j\in U_j}|\phi^\lambda_{j\overline{v_1}...\overline{v_q}}(x_j)|$$

We state some estimates useful in our proof:

1. $\forall k\geq 2,\forall\phi\in\mathscr{L}^{0,q}(T),\exists c_1\in\mathbb{R}$ depending only on k and α such that

$$|\phi|_{k+\alpha}\leq c_1(|\square\phi|_{k-2+\alpha}+|\phi|_0)$$

2. $\forall k \geq 2, \forall \psi \in \mathscr{L}^{0,2}(T), \exists c_2 \in \mathbb{R}$ independent of ψ such that

$$|G\psi|_{k+\alpha} \leq c_2 |\psi|_{k-2+\alpha}$$

3. $\exists k_1, k_2 \in \mathbb{R}$ independent of forms concerned such that

$$|\vartheta\psi|_{k+\alpha} \leq k_1 |\psi|_{k+1+\alpha}, \qquad \forall \psi \in \mathscr{L}^{0,2}(T),$$
$$|[\phi, \psi]|_{k-1+\alpha} \leq k_2 |\phi|_{k+\alpha} |\psi|_{k+\alpha}, \qquad \forall \phi, \psi \in \mathscr{L}^{0,1}(T).$$

We follow the notation that $P(t) \ll Q(t)$ if $P(t) = \sum P_{v_1 \dots v_m} t_1^{v_1} \dots t_m^{v_m}, Q(t) = \sum Q_{v_1 \dots v_m} t_1^{v_1} \dots t_m^{v_m}$ and $|P_{v_1 \dots v_m}| \leq |Q_{v_1 \dots v_m}|$ in each index, and define $|P|_{k+\alpha}(t) = \sum |P_{v_1 \dots v_m}|_{k+\alpha} t_1^{v_1} \dots t_m^{v_m}$.

We need an absolute convergent series $A(t)$ in Δ_ε so that for the constructed $\phi(t)$, we have $|\phi|_{k+\alpha} \ll A(t)$. The candidate for such a choice is

$$A(t) = \frac{b}{16c} \sum_{v=1}^{\infty} \frac{(t_1 + \dots + t_m)^v}{v^2}, \tag{3.5}$$

for some constants $b, c > 0$. By a simple argument of "AM $\geq$ GM", one can show that $A(t)^2 \ll \frac{b}{c} A(t)$.

Lemma 3.6 *Fix $k \geq 2$, then $\forall v \geq 1$, $\exists b, c$ sufficiently large so that $|\phi^{[v]}|_{k+\alpha}(t) \ll A(t)$. In other words, $|\phi|_{k+\alpha}(t) \ll A(t)$.*

Proof of Lemma 3.6

We proceed by induction on v.

- $v = 1$: Here $\phi^{[1]}(t) = \beta_1 t_1 + \dots + \beta_m t_m$ and the linear term of $A(t)$ is $\frac{b}{16}(t_1 + \dots + t_m)$. Hence, we can choose b such that $|\beta_\lambda|_{k+\alpha} < \frac{b}{16}$.
- Assume the statement is true for some v. Then for some k_1, k_2 independent

of $\phi(t)$ and $\psi(t)$,

$$\begin{aligned}
&|\phi_{[v+1]}|_{k+\alpha}(t) = |\vartheta G\psi_{[v+1]}|_{k+\alpha}(t) \ll k_1|G\psi_{[v+1]}|_{k+1+\alpha}(t) \ll k_1c_1|\psi_{[v+1]}|_{k-1+\alpha}(t) \\
&\ll k_1c_1\big(k_2|\phi^{[v]}|_{k+\alpha}(t)|\phi^{[v]}|_{k+\alpha}(t)\big) \qquad (\because \psi_{[v+1]}(t) = \frac{1}{2}[\phi^{[v]}(t), \phi^{[v]}(t)]) \\
&\ll k_1k_2c_1A(t)^2 \qquad (\because \text{ Induction hypothesis}) \\
&\ll \frac{k_1k_2c_1b}{c}A(t)
\end{aligned}$$

By putting $c = k_1k_2c_1b$ and observing $\phi^{[v+1]} = \phi^{[v]} + \phi_{[v+1]}$, we are done. ■

By **Lemma 3.6**, since $A(t)$ converges absolutely on Δ_ε, $0 < \varepsilon \leq \frac{1}{mc}$, $\phi(t) = \sum\phi_{[v]}(t)$ converges with respect to $|\cdot|_{k+\alpha}, \forall t \in \Delta_\varepsilon$. This implies that $\phi(t)$ is a C^{k} vector (0,1)-form on $M \times \Delta_\varepsilon$. Nevertheless, this does **NOT** immediately imply $\phi(t)$ is C^∞, as the estimate constant depends on k. To achieve this goal, one should observe that $\phi(t)$ is the solution to the following quasi-linear elliptic equation:

$$\Big[-\sum_{\lambda=1}^{m}\frac{\partial^2}{\partial t_\lambda\partial\bar{t}_\lambda} + \Box\Big]\phi(t) - \frac{1}{2}\vartheta[\phi(t), \phi(t)] = 0 \tag{3.6}$$

By some a priori estimate of the solutions to (3.6), one can prove that $\phi(t)$ is C^∞ ([6], p.452-8).

Chapter 4

Completeness Theorem

4.1 The Completeness Theorem

Definition 4.1 *Let $(\mathcal{M}, B, \varpi)$ be a complex analytic family of compact complex manifolds, then it is* **complete at** $t^0 \in B$ *if for any complex analytic family $(\mathcal{N}, D, \pi)$, with $0 \in D$ is a domain in $\mathbb{C}^l$, $\pi^{-1}(0) = \varpi^{-1}(t^0)$, there always exists a sufficiently small domain E with $0 \in E \subseteq D$ and holomorphic $h : s \to t = h(s)$ with $h(0) = t^0$ such that $(\mathcal{N}_E, E, \pi)$ is induced from $(\mathcal{M}, B, \varpi)$ by h where $\mathcal{N}_E = \pi^{-1}(E)$. $(\mathcal{M}, B, \varpi)$ is called a complete complex analytic family if it is complete at each $h \in B$.*

REMARK If $(\mathcal{M}, B, \varpi)$ is complete at $t^0 \in B$, then it contains all deformations N_s of M_{t^0} provided that s is sufficiently small.

The notion of completeness helps clarifying the concept of the number of moduli and effective parameters (§**4.4**). In this section, we first consider the following fundamental question: *When does a family $(\mathcal{M}, B, \pi)$ become complete?*

Theorem 4.1 *Given a family $(\mathcal{M}, B, \varpi)$ with $M = M_0 = \varpi^{-1}(0)$, if $\rho_0 : T_0(B) \to H^1(M_0, \Theta_0)$ (Recall: $\rho_0(\frac{\partial}{\partial t}) := \frac{\partial M_t}{\partial t}|_{t=0}$) is surjective, then $(\mathcal{M}, B, \varpi)$ is complete at $0 \in B$.*

Lemma 4.1 *1. If $(\mathcal{N}_\Delta, \Delta, \pi)$ is induced from $(\mathcal{M}, B, \varpi)$ by h, $h(0) = 0$, then the restriction of the projection $\mathcal{M} \times \Delta \to \mathcal{M}$ to $\mathcal{N}_\Delta$, denoted as g, maps each N_s biholomorphically onto $M_{h(s)}$, and g is the extension of the $g_0 : N_0 \to M_0$.*

2. If the identity $g_0 : M_0 \to N_0$ can be extended to a holomorphic $g : \mathcal{N}_\Delta \to \mathcal{M}$ such that for each $s \in \Delta$, g maps each N_s biholomorphically onto $M_{h(s)}$, then $(\mathcal{N}_\Delta, \Delta, \pi)$ is the complex analytic family induced from $(\mathcal{M}, B, \varpi)$ by h.

PROOF OF LEMMA 4.1

1. $\because \forall s \in \Delta,\ N_s = \pi^{-1}(s) = M_{h(s)} \times s$
   $\therefore \forall (p, s) \in N_s = M_{h(s)} \times s,\ g(p, s) = p \in M_{h(s)}$
   Further, $N_0 = M_{h(0)} \times 0 = M_0 \times 0 \Rightarrow g(N_0) = M_0$

2. Let $(\hat{\mathcal{N}}, \Delta, \hat{\pi})$ induced from $(\mathcal{M}, B, \varpi)$ by $h : \Delta \to B$, then define $\Phi : \mathcal{N}_\Delta \to \mathcal{M} \times \Delta$ by $\Phi(q) = (g(q), \pi(q)), \forall q \in \mathcal{N}_\Delta$.

$$\Phi(N_s) = \Phi(M_{h(s)} \times s) = (g(M_{h(s)} \times s), \pi(M_{h(s)} \times s)) = M_{h(s)} \times s,$$

   mapping $\mathcal{N}_\Delta = \cup_{s\in\Delta} N_s$ biholomorphically onto $\hat{\mathcal{N}} = \cup_{s\in\Delta} M_{h(s)} \times s$. Further, $\hat{\pi}\Phi(N_s) = \hat{\pi}(M_{h(s)} \times s) = s = \pi(N_s)$. Hence, $(\mathcal{N}_\Delta, \Delta, \pi)$ and $(\hat{\mathcal{N}}, \Delta, \hat{\pi})$ are biholomorphically equivalent. ■

From the above lemma, one can prove **Theorem 4.1** by constructing holomorphic $h : \Delta \to B$ and holomorphic $g : \mathcal{N}_\Delta \to \mathcal{M}$ such that $h(0) = 0$ and g maps each N_s biholomorphically onto $M_{h(s)}$.

4.2 Construction of Formal Power Series

Some notations on local coodinates are first fixed for the following presentation:

$$\begin{cases} \mathcal{M} = \cup_j \mathcal{U}_j, & \mathcal{U}_j = \{(\zeta_j, t) \in \mathbb{C}^n \times B \big| \ |\zeta_j| < 1\} \quad \zeta_j = g_{jk}(\zeta_k, t) \text{ on } U_j \cap U_k \\ \mathcal{N} = \cup_j \mathcal{W}_j, & \mathcal{W}_j = \{(z_j, t) \in \mathbb{C}^n \times D \big| \ |z_j| < 1\} \quad z_j = f_{jk}(z_k, t) \text{ on } W_j \cap W_k \\ B = \{t \in \mathbb{C}^m \big| \ |t| < 1\} D = \{s \in \mathbb{C}^l \big| \ |s| < 1\} \end{cases}$$

Since given $M_0 = N_0$, we can assume $\mathcal{W}_j \cap N_0 = \mathcal{U}_j \cap M_0$ and $(\zeta_j, 0)$ and $(z_j, 0)$ coincide on $\mathcal{W}_j \cap N_0 = \mathcal{U}_j \cap M_0 = U_j$

To prove the completeness theorem, it suffices to construct holomorphic map h on some Δ_ε with $t = h(s), h(0) = 0$; and extend $g_0 : N_0 \to M_0$ to some holomorphic $g : \pi^{-1}(\Delta_\varepsilon) \to \mathcal{M}$ such that $\varpi \circ g = h \circ \pi$.

Lemma 4.2 *Suppose h and g are so constructed, then there exists a continuous function $\varepsilon(z_j)$ on U_j with $0 < \varepsilon(z_j) \leq \varepsilon$ such that*

$$\mathcal{W}_j^* := \{(z_j, s) \big| \ |z_j| < 1, |s| < \varepsilon(z_j)\} \subseteq g^{-1}(\mathcal{U}_j) \cap \mathcal{W}_j, \quad \mathcal{W}_j^* \supseteq U_j$$

and there exists holomorphic $g_j(z_j, s) = (g_j^1(z_j, s), ..., g_j^n(z_j, s))$ on $\mathcal{W}_j^*$ such that

$$g(z_j, s) = (\zeta_j, t) = (g_j(z_j, s), h(s)), \qquad \forall (z_j, s) \in \mathcal{W}_j^* \tag{4.1}$$

$$g_j(f_{jk}(z_j, s), s) = g_{jk}(g_k(z_k, s), h(s)), \qquad \forall (z_k, s) \in \mathcal{W}_k^* \cap \mathcal{W}_j^* \tag{4.2}$$

Proof of Lemma 4.2

Since $\forall U_j \subseteq N_0$, $g(U_j) = U_j \subseteq \mathcal{U}_j$, $g^{-1}(\mathcal{U}_j)$ is open, containing U_j and hence $\varepsilon(z_j)$ and $\mathcal{W}_j^*$ can be so constructed. Under this setting, $g(\mathcal{W}_j^*) \subseteq \mathcal{U}_j$ and thus (4.1) can also be achieved. Now, $\forall (z_k, s) \in \mathcal{W}_k^* \cap \mathcal{W}_j^*$,

$$g_j(f_{jk}(z_k, s), s) = g_j(z_j, s) = \zeta_j = g_{jk}(\zeta_k, t) = g_{jk}(g_k(z_k, s), h(s)) \ \blacksquare$$

Note that since $g(z_j, 0) = (z_j, 0)$, we have $g_j(z_j, 0) = z_j$ and $h(0) = 0$. Hence, by **Lemma 4.2**, we are to construct formal power series

$$g_j(z_j, s) = z_j + \sum_v g_{j|v}(z_j, s), \qquad h(s) = \sum_v h_{[v]}(s)$$

such that (4.2) holds by substituting the power series. Notations on the partial sum:

$$h^{[v]}(s) = h_{[1]}(s) + ... + h_{[v]}(s); \qquad g_j^{[v]}(z_j, s) = z_j + g_{j|1}(z_j, s) + ... + g_{j|v}(z_j, s)$$

Lemma 4.3 (4.2) *is equivalent to*

$$g_j^{[v]}(f_{jk}(z_k, s), s) \equiv_v g_{jk}(g_k^{[v]}(z_k, s), h^{[v]}(s)), \qquad v = 0, 1, 2, ... \tag{4.3}$$

PROOF OF LEMMA 4.3

For $v = 0$, (4.3) is equivalent to $f_{jk}(z_k, 0) = g_{jk}(z_k, 0)$, which clearly holds.
For $v \geq 1$, (4.2) leads to a relation

$$f_{jk}(z_k, s) + \sum_v g_{j|v}(f_{jk}(z_k, s), s) = g_{jk}\Big(z + \sum_v g_{k|v}(z_k, s), \sum_v h_{[v]}(s)\Big),$$

whose terms of degree $\leq m$ involves no $g_{j|v}(\cdot, s), g_{k|v}(\cdot, s)$ and $h_{[v]}(s)$ with $v > m$. Thus, (4.2) is also equivalent to (4.3) in this case. ∎

Lemma 4.4 *Writing* $h_{[v]}(s) = (h_{1|v}(s), ..., h_{m|v}(s))$, (4.3) *is equivalent to the congruence*

$$\Gamma_{jk|v}(z_j, s) \equiv_v \sum_\beta \frac{\partial z_j}{\partial z_k^\beta} \cdot g_{k|v}^\beta(z_k, s) - g_{j|v}(z_j, s) + \sum_r \left(\frac{\partial g_{jk}(z_k, t)}{\partial t_r}\right)\Big|_{t=0} h_{r|v}(s) \tag{4.4}$$

where $\Gamma_{jk|v}(z_j, s)$ *is the sum of terms of degree* v *of the expression* $g_j^{[v-1]}(f_{jk}(z_k, s), s) - g_{jk}(g_k^{[v-1]}(z_k, s), h^{[v-1]}(s))$

PROOF OF LEMMA 4.4

$$\begin{aligned} \because g_j^{[v]}(f_{jk}(z_k, s), s) &= g_j^{[v-1]}(f_{jk}(z_k, s), s) + g_{j|v}(f_{jk}(z_k, s), s) \\ &\equiv_v g_j^{[v-1]}(f_{jk}(z_k, s), s) + g_{j|v}(z_j, s) \end{aligned}$$

$$
\begin{aligned}
&\therefore\, g_{jk}(g_k^{[v]}(z_k,s),h(s)) \\
&= g_{jk}\big(g_k^{[v-1]}(z_k,s)+g_{k|v}(z_k,s),h^{[v-1]}(s)+h_{[v]}(s)\big) \\
&\equiv_v g_{jk}(g_k^{[v-1]}(z_k,s),h^{[v-1]}(s))+\sum_{\beta}^{n}\frac{\partial g_{jk}}{\partial\zeta_k^\beta}(g_k^{[v-1]}(z_k,s),h^{[v-1]}(s))g_{k|v}^\beta(s) \\
&\qquad +\sum_{r}^{m}\frac{\partial g_{jk}}{\partial t_r}(g_k^{[v-1]}(z_k,s),h^{[v-1]}(s))h_{r|v}(s) \\
&\equiv_v g_{jk}(g_k^{[v-1]}(z_k,s),h^{[v-1]}(s))+\sum_{\beta}^{n}\frac{\partial g_{jk}}{\partial\zeta_k^\beta}(z_k,0)g_{k|v}^\beta(s)+\sum_{r}^{m}\frac{\partial g_{jk}}{\partial t_r}(z_k,0)h_{r|v}(s) \\
&\equiv_v g_{jk}(g_k^{[v-1]}(z_k,s),h^{[v-1]}(s))+\sum_{\beta}^{n}\frac{\partial z_j}{\partial\zeta_k^\beta}g_{k|v}^\beta(s)+\sum_{r}^{m}\Big(\frac{\partial g_{jk}(z_k,t)}{\partial t_r}\Big)\Big|_{t=0}h_{r|v}(s)
\end{aligned}
$$

$\therefore$ We are done. ■

Now, introduce the following holomorphic vector fields:

$$
\begin{aligned}
\theta_{rjk} &= \sum_{\alpha}^{n}\Big(\frac{\partial g_{jk}^\alpha(z_k,t)}{\partial t_r}\Big)_{t=0}\frac{\partial}{\partial z_j^\alpha} \\
\Gamma_{jk|v}(s) &= \sum_{\alpha}^{n}\Gamma_{jk|v}^\alpha(z_j,s)\frac{\partial}{\partial z_j^\alpha} \\
g_{k|v}(s) &= \sum_{\beta}^{n}g_{k|v}^\beta(z_k,s)\frac{\partial}{\partial z_k^\beta}
\end{aligned}
$$

Then (4.4) can be rewritten as

$$
\{\Gamma_{jk|v}(s)\}=\sum_{r}^{m}h_{r|v}(s)\{\theta_{rjk}\}+\delta\{g_{j|v}(s)\} \qquad v\geq 1 \text{ in } Z^1(\mathcal{U},\Theta_0) \tag{4.5}
$$

Note that $\{\theta_{rjk}\}$ represents the infinitesimal deformation $\theta_r=\rho_0\Big(\frac{\partial}{\partial t_r}\Big)\in H^1(M_0,\Theta_0)$. Now, we can conclude that solving (4.2) by power series expansion is equivalent to solving $h_{r|v}(s)$ and $\{g_{j|v}(s)\}$ for each $v\geq 1$ in (4.5).

Lemma 4.5 *1. If there exists $v\geq 1, h_{r|v}(s), r=1,...,m, \{g_{j|v(s)}\}$ so that (4.5) is solved for that index v, then $\{\Gamma_{jk|v}(s)\}$ is a 1-cocycle.*

2. If ρ_0 is surjective and $\{\Gamma_{jk|v}(s)\}$ forms a 1-cocycle for some $v\geq 1$, then (4.5) has solution $h_{r|v}(s), r=1,...,m, \{g_{j|v}(s)\}$ for that v.

Proof of Lemma 4.5

1. Obvious from (4.5), and $\{\theta_{rjk}\}, \delta\{g_{j|v}\} \in Z^1(\mathcal{U}, \Theta_0)$.

2. It suffices to prove that the coefficients $\{\Gamma_{jk}\}$ of any $\{\Gamma_{jk|v}(s)\} \in Z^1(\mathcal{U}, \Theta_0)$ can be written as $\{\Gamma_{jk}\} = \sum_r^m h_r\{\theta_{rjk}\} + \delta\{g_j\}, h_r \in \mathbb{C}, \{g_j\} \in C^0(\mathcal{U}, \Theta_0)$.

   Let $\gamma \in H^1(M_0, \Theta_0)$ be the cohomology class of $\{\Gamma_{jk}\}$, then since $\rho_0 : T_0(B) \to H^1(M_0, \Theta_0)$ is surjective, $\exists h_r \in \mathbb{C}$ such that $\gamma = \sum_r^m h_r\theta_r \Rightarrow \{\Gamma_{jk}\} - \{\sum_r^m h_r\theta_{rjk}\} = 0$ in $H^1(M_0, \Theta_0)$. Finally, by the natural inclusion $H^1(\mathcal{U}, \Theta_0) \hookrightarrow H^1(M_0, \Theta_0)$, $\{\Gamma_{jk}\} - \sum_r^m h_r\{\theta_{rjk}\} \in \delta C^0(\mathcal{U}, \Theta_0)$. ■

Lemma 4.6 *If ρ_0 is surjective, then we can construct $h(s)$ and $g_j(z_j, s)$ such that (4.2) is solved.*

Proof of Lemma 4.6

(For convenience, some z_k's are dropped when there is no ambiguity by so doing) By **Lemma 4.5 2.** and the remark just beneath the proof of **Lemma 4.4**, it suffices to show that $\{\Gamma_{jk|v}(s)\} \in Z^1(\mathcal{U}, \Theta_0)$.

$$\because \Gamma_{ik|v}(z_i, s) \equiv_v g_i^{[v-1]}(f_{ik}(s), s) - g_{ik}(g_k^{[v-1]}(s), h^{[v-1]}(s))$$

$$= g_i^{[v-1]}(f_{ik}(s), s) - g_{ij}(g_{jk}(g_k^{[v-1]}(s), h^{[v-1]}(s)), h^{[v-1]}(s))$$

$$(\because g_{ik}(\zeta_k, t) = g_{ij}(g_{jk}(\zeta_k, t), t))$$

$$\equiv_v g_i^{[v-1]}(f_{ik}(s), s) - g_{ij}(g_j^{[v-1]}(f_{jk}(s), s) - \Gamma_{jk|v}(z_j, s), h^{[v-1]}(s))$$

$$\equiv_v g_i^{[v-1]}(f_{ik}(s), s) - g_{ij}(g_j^{[v-1]}(f_{jk}(s), s), h^{[v-1]}(s)) - \sum_\beta^n \frac{\partial g_{ij}}{\partial \zeta_j^\beta}(g_j^{[v-1]}(f_{jk}(s), s), h^{[v-1]}(s))\Gamma_{jk|v}^\beta(z_j, s)$$

$$\equiv_v g_i^{[v-1]}(f_{ik}(s), s) - g_{ij}(g_j^{[v-1]}(f_{jk}(s), s), h^{[v-1]}(s)) - \sum_\beta^n \frac{\partial z_i}{\partial z_j^\beta}\Gamma_{jk|v}^\beta(z_j, s)$$

$$(\because \frac{\partial g_{ij}}{\partial \zeta_j^\beta}(g_j^{[v-1]}(f_{jk}(0), 0), 0) = \frac{\partial g_{ij}}{\partial \zeta_j^\beta}(z_j, 0) = \frac{\partial z_i}{\partial z_j^\beta})$$

$$\equiv_v \Gamma_{ij|v}(z_i, s) - \sum_{\beta}^{n} \frac{\partial z_i}{\partial z_j^\beta} \Gamma^\beta_{jk|v}(z_j, s)$$

$$\Rightarrow \Gamma_{ik|v}(s) - \Gamma_{ij|v}(s) + \Gamma_{jk|v}(s) = 0 \blacksquare$$

4.3 Convergence Proof

In §4.2, formal solutions $h(s)$ and $g_j(z_j, s)$ were constructed. In this section, we are to show there are $h_{r|v}(s)$ and $\{g_{j|v}(z_j, s)\}$ so that the series converge. From (4.4), one has

$$\Gamma_{jk}(z_j) = \sum_{r}^{m} h_r \theta_{rjk}(z_j) + \sum_{\alpha,\beta} \frac{\partial z_j^\alpha}{\partial z_k^\beta} g_k^\beta(z_k) - g_j(z_j) \tag{4.6}$$

Denote $\{\Gamma_{jk}\}$ by Γ and define $|\Gamma| = \max_{j,k} \sup_{z_j \in U_j \cap U_k} |\Gamma_{jk}(z_j)|$.

Lemma 4.7 *There exists $h_r, \{g_j\}$ solutions to (4.6) satisfying*

$$|h_r| \leq K|\Gamma|, \qquad |g_j(z_j)| \leq K|\Gamma|$$

for some constant K independent of Γ.

PROOF OF LEMMA 4.7

From the definition of Γ, we have $|\Gamma| < +\infty$. It suffices to prove that there is a constant K such that

$$\inf \max_{r,j} \{|h_r|, \sup_{z_j \in U_j} |g_j(z_j)|\} \leq K|\Gamma|,$$

where the infimum is taken over all solutions $h_r, g_j(z_j)$

Suppose otherwise, i.e. $\exists \Gamma^{(v)} = \{\Gamma^{(v)}_{jk}\} \in Z^1(\mathfrak{U}, \Theta_0)$, and correspondingly $h^{(v)}, \{g_j^{(v)}\}$ such that $|h_r^{(v)}| < 2$, $|g_j^{(v)}(z_j)| < 2$, $|\Gamma^{(v)}| < \frac{1}{v}$; and for ALL solutions, we have $|h_r^{(v)}|, \sup_{z_j \in U_j} |g_j(z_j)| \geq 1$.

Now, passing to subsequences, assume that $\{g_j^{(v)}(z_j)\}$ converges in U_j, and for all r, $\{h_r^{(v)}\}$ converges.Moreover, the former should converge uniformly on U_j, from the relation (4.6) (In particular, $|\Gamma_{jk}^{(v)}(z_j)| \leq |\Gamma^{(v)}| \to 0$, $\{h_r^{(v)}\}$ converges and all other terms are bounded on $U_j \cap U_k$).

Denote $h_r = \lim_v h_r^{(v)}$, $g_j(z_j) = \lim_v g_j^{(v)}(z_j)$, then $H_r^{(v)} := h_r^{(v)} - h_r$, $G_j^{(v)}(z_j) := g_j^{(v)}(z_j) - g_j(z_j)$ constitute another solution set of (4.6) for $\Gamma^{(v)}$. However, the fact that $H^{(v)} \to 0$, $\sup_{z_j \in U_j} |G_j^{(v)}(z_j)| \to 0$ contradicts with our assumption. ■

We now make use of the majorant series to prove the convergence. First of all, repeated use of $A(s)^2 \ll \frac{b}{c}A(s)$, $A(s)$ as defined in (3.5) yields

$$A(s)^v \ll \left(\frac{b}{c}\right)^{v-1} A(s), \qquad v = 1, 2, 3, ... \tag{4.7}$$

It suffices to prove the following

Theorem 4.2 *There exists sufficiently large b and c such that $h^{[v]}(s) \ll A(s)$ and $g_j^{[v]}(z_j, s) - z_j \ll A(s)$ for all $v \geq 1$.*

PROOF OF THEOREM 4.2

Proceed by induction on v. $v = 1$ can be achieved by simply choosing b sufficiently large. Assume the estimates are true for some $v - 1 \geq 0$, i.e. by writing

$$g_j^{[v-1]}(z_j, s) = \sum_{v_1 ... v_l} g_{jv_1...v_l}^{[v-1]}(z_j) s_1^{v_1} ... s_l^{v_l}$$

$$g_{jv_1...v_l}^{[v-1]}(z_j + \zeta) = \sum_{\mu_1 ... \mu_n} g_{jv_1...v_l\mu_1...\mu_n}^{[v-1]}(z_j) \zeta_1^{\mu_1} ... \zeta_n^{\mu_n}$$

$$A(s) = \sum_{v_1 ... v_l} A_{v_1...v_l} s_1^{v_1} ... s_l^{v_l}$$

we have $h^{[v-1]}(s) \ll A(s)$,
and $|g_{jv_1...v_l}^{[v-1]}(z_j)|$, $|g_{jv_1...v_l}^{[v-1]}(z_j + \zeta)| < A_{v_1...v_l}$, for all $|\zeta|$ small, $v_1 + ... v_l \geq 1$.

Since $f_{jk}(z_k, s) = z_j + \sum_v f_{jk|v}(z_j, s)$ is given, we can assume

$$\zeta := f_{jk}(z_k, s) - z_j \ll A_0(s), \qquad A_0(s) := \frac{b_0}{16c_0}\sum_{v=1}^{\infty}\frac{c_0^v(s_1 + \dots + s_l)^v}{v^2} \tag{4.8}$$

$$\begin{aligned}
&\therefore g_j^{[v-1]}(z_j+\zeta, s) - g_j^{[v-1]}(z_j, s) - f_{jk}(z_k, s) + z_j \\
&= \Big(g_j^{[v-1]}(z_j+\zeta, s) - (z_j+\zeta)\Big) - \Big(g_j^{[v-1]}(z_j, s) - z_j\Big) \\
&= \sum_{v_1+\dots v_l \geq 1} g^{[v-1]}_{jv_1\dots v_l}(z_j+\zeta)s_1^{v_1}\dots s_l^{v_l} - \sum_{v_1+\dots v_l \geq 1} g^{[v-1]}_{jv_1\dots v_l}(z_j)s_1^{v_1}\dots s_l^{v_l} \\
&= \sum_{v_1+\dots v_l \geq 1}\Big(g^{[v-1]}_{jv_1\dots v_l}(z_j+\zeta) - g^{[v-1]}_{jv_1\dots v_l}(z_j)\Big)s^{v_1}\dots s_l^{v_l} \\
&= \sum_{v_1+\dots v_l \geq 1}\Big(\sum_{\mu_1+\dots\mu_n\geq 1} g_{jv_1\dots v_l\mu_1\dots\mu_n}(z_j)\zeta_1^{\mu_1}\dots\zeta_n^{\mu_n}\Big)s_1^{v_1}\dots s_l^{v_l} \\
&= \sum_{v_1+\dots+v_l \geq 1}\Big(\sum_{\mu_1+\dots\mu_n\geq 1}\big(\frac{1}{2\pi i}\big)^n\int_{|\eta_1|,\dots,|\eta_n|=\delta}\frac{g_{jv_1\dots v_l}(z_j+\eta)d\eta_1\dots d\eta_n}{\eta_1^{\mu_1+1}\dots\eta_n^{\mu_n+1}}\zeta_1^{\mu_1}\dots\zeta_n^{\mu_n}\Big)s_1^{v_1}\dots s_l^{v_l} \\
&\ll \sum_{v_1+\dots+v_l \geq 1}\Big[\sum_{\mu_1+\dots+\mu_n\geq 1}\frac{A_{v_1\dots v_l}}{\delta^{\mu_1+\dots+\mu_n}}\zeta_1^{\mu_1}\dots\zeta_n^{\mu_n}\Big]s_1^{v_1}\dots s_l^{v_l} \qquad (\because \text{ Induction hypothesis}) \\
&\ll \sum_{\mu_1+\dots+\mu_n\geq 1} A(s)\frac{\zeta_1^{\mu_1}\dots\zeta_n^{\mu_n}}{\delta^{\mu_1+\dots+\mu_n}} \ll \sum_{\mu_1+\dots+\mu_n\geq 1} A(s)\Big(\frac{A_0(s)}{\delta}\Big)^{\mu_1+\dots+\mu_n} \qquad (\text{From (4.8)}) \\
&= A(s)\Big[\Big(\sum_{\mu=0}^{\infty}\big(\frac{A_0(s)}{\delta}\big)^{\mu}\Big)^n - 1\Big] = A(s)\Big[\Big(1+\sum_{\mu=1}^{\infty}\big(\frac{A_0(s)}{\delta}\big)^{\mu}\Big)^n - 1\Big] \\
&\leq A(s)\Big[\Big(1+\frac{1}{\delta}\sum_{\mu=1}^{\infty}\big(\frac{b_0}{c_0\delta}\big)^{\mu-1}A_0(s)\Big)^n - 1\Big] \qquad (\text{From (4.7)}) \\
&\ll A(s)\big[\big(1+\frac{2}{\delta}A_0(s)\big)^n - 1\big], \qquad \text{by choosing } c_0 \text{ large enough so that } \frac{b_0}{c_0\delta} < \frac{1}{2} \\
&\ll A(s)\Big[\sum_{k=1}^{n} C_k^n\Big(\frac{2}{\delta}\Big)^k A_0(s)^k\Big] \ll A(s)\Big[\frac{2}{\delta}\sum_{k=1}^{n} C_k^n\big(\frac{2b_0}{c_0\delta}\big)^{k-1}A_0(s)\Big] \qquad (\text{From (4.7)}) \\
&\ll \frac{2}{\delta}\sum_{k=0}^{n} C_k^n A_0(s) = \frac{2^{n+1}}{\delta}A_0(s)A(s) \ll \frac{2^{n+1}}{\delta}\Big(\frac{b_0}{b}A(s)\Big)A(s), \\
&\qquad\qquad \text{by taking } b, c \text{ such that } b > b_0, c > c_0 \Rightarrow A_0(s) \ll \frac{b_0}{b}A(s) \\
&\ll \frac{2^{n+1}b_0}{\delta b}\Big(\frac{b}{c}A(s)\Big) = \frac{2^{n+1}b_0}{c\delta}A(s)
\end{aligned}$$

$$\begin{aligned}
&\therefore g_j^{[v-1]}(f_{jk}(z_k,s),s) - g_j^{[v-1]}(z_j,s)\\
&\ll \frac{2^{n+1}b_0}{c\delta}A(s) + f_{jk}(z_k,s) - z_j \ll \frac{2^{n+1}b_0}{c\delta}A(s) + A_0(s)\\
&\ll \Big(\frac{2^{n+1}b_0}{c\delta} + \frac{b_0}{b}\Big)A(s)
\end{aligned}$$

On the other hand, since $g_{jk}(z_k,t)$ is given, we can assume that

$$g_{jk}(z_k+\zeta,t) - z_t - L(\zeta,t) \ll \sum_{\mu=2}^{\infty} a_0^{\mu}(\zeta_1+\ldots+\zeta_n+t_1+\ldots+t_m)^{\mu},$$

where $L(\zeta,t)$ is the linear term of the expansion of $g_{jk}(z_k+\zeta,t)$ in ζ. Then by the induction hypothesis, $\zeta = g_k^{[v-1]}(z_k,s) - z_k \ll A(s)$, $t = h^{[v-1]}(s) \ll A(s)$, and

$$\begin{aligned}
&\big[g_{jk}(g_k^{[v-1]}(z_k,s), h^{[v-1]}(s))\big]_{[v]}\\
&\ll \sum_{\mu=2}^{\infty} a_0^{\mu}(m+n)^{\mu}A(s)^{\mu} \ll \sum_{\mu=2}^{\infty} a_0^{\mu}(m+n)^{\mu}\Big(\frac{b}{c}\Big)^{\mu-1}A(s)\\
&= \frac{ba_0^2(m+n)^2}{c}\sum_{\mu=0}^{\infty}\Big(\frac{ba_0(m+n)}{c}\Big)^{\mu}A(s) \ll \frac{2ba_0^2(m+n)^2}{c}A(s),
\end{aligned}$$

by choosing large c such that $\frac{bc_0(m+n)}{c} < \frac{1}{2}$. Hence by the definition of $\Gamma_{jk|v}(z_j,s)$ (**Lemma 4.4**), we have

$$\Gamma_{jk|v}(z_j,s) \ll \Big[\frac{2^{n+1}b_0}{c\delta} + \frac{b_0}{b} + \frac{2ba_0^2(m+n)^2}{c}\Big]A(s)$$

Essentially, by **Lemma 4.7**, this implies that there exists solutions $h_{r|v}(s), \{g_{j|v}(s)\}$ such that $h_{r|v}(s), g_{j|v}(s) \ll A(s)$, by first choosing b large and then c so that $\frac{c}{b} \gg 1$. Finally, since $h^{[v]}(s) = h^{[v-1]}(s) + h_{[v]}(s), g_j^{[v]}(z_j,s) = g_j^{[v-1]}(z_j,s) + g_{j|v}(z_j,s)$, it suffices to conclude the proof. ■

Proof of the Completeness Theorem

Restricting to $\Delta = \{s \in \mathbb{C}^l \mid |s| < \varepsilon\}, \varepsilon > 0$ so that $h(s)$ converges absolutely and $g_j(z_j,s)$ converges absolutely and uniformly for $z_j \in U_j$, as implied by **Lemma**

4.7. Without loss of generality, assume $\mathcal{N}_\Delta = \cup_j U_j \times \Delta, \mathcal{M} = \cup_j U_j \times B$, where $U_j = \mathcal{W}_j \cap N_0 = \mathcal{U}_j \cap M_0$ as described in §4.2. Now, the map $g_j(z_j, s) = (g_j(z_j, s), h(s))$ constructed in §4.2 has the following properties:

- $g_j(U_j \times \Delta) \subseteq \mathbb{C}^n \times B$. In fact, we can assume $g_j(U_j \times \Delta) \subseteq U_j \times B \subseteq \mathcal{M}$ by further restriction on g_j.
- $g_j\big|_{U_j} = \mathrm{id}_{U_j}$, i.e., $g_j(z_j, 0) = (z_j, 0)$
- $\forall c \in U_k \cap U_j, \exists \mathcal{V}(c) := \{(z_k, s)|\ |z_k - c| < \varepsilon, |s| < \varepsilon\} \subseteq g_k^{-1}(U_k \times B) \cap g_j^{-1}(U_j \times B)$ such that $g_j = g_k$ on $\mathcal{V}(c)$. But this is essentially the relation (4.2) for all $(z_k, s) \in \mathcal{V}(c)$.

Therefore, there exists a holomorphic $g : \cup_j U_j \times \Delta \to \mathcal{M}$ such that $g\big|_{U_j \times \Delta} = g_j$. Moreover, $g(z_j, s) = (g_j(z_j, s), h(s)), \forall (z_j, s) \in U_j \times \Delta$, implying that $\varpi \circ g = h \circ \pi$. Finally, $g_j(z_j, 0) = z_j$ implies that g is an extension of the identity $g_0 : N_0 \to M_0$.
■

4.4 Effective Parameters and Number of Moduli

Given a complex analytic family $(\mathcal{M}, B, \varpi), B \subseteq \mathbb{C}^m$, and suppose that $\rho_t : \frac{\partial}{\partial t} \to \frac{\partial M_t}{\partial t}$ is injective, then the complex structure of $M_t \in \mathcal{M}$ varies with t and we have

Definition 4.2 *$t = (t_1, ..., t_m)$ is an **effective parameter** for $(\mathcal{M}, B, \varpi)$ if $\rho_t : T_t(B) \to H^1(M_t, \Theta_t)$ is injective for each $t \in B$. In this case, $(\mathcal{M}, B, \varpi)$ is called an **effectively parametrized** complex analytic family.*

Suppose an effectively parametrized family $(\mathcal{M}, B, \varpi)$ is given with $\varpi^{-1}(0) = M$, then $\dim\ \rho_0(T_0(B)) = m$. Now, consider an arbitrary θ in $H^1(M, \Theta)$. If it happens that $\exists c_\lambda \in \mathbb{C}$ such that $\theta = \rho_0\big(\sum c_\lambda \frac{\partial}{\partial t_\lambda}\big)$, then by a holomorphic map

$h : \mathbb{C} \to \mathbb{C}^m$ defined by $h(s) = (c_1 s, ..., c_m s)$, the induced family $\{M_{h(s)} | |s| < \varepsilon\}$, $\varepsilon > 0$ sufficiently small, has the following properties:

1. $M_{h(0)} = M_0 = M$;
2. $\left(\frac{dM_{h(s)}}{ds}\right)_{s=0} = \sum c_\lambda \left(\frac{\partial M_t}{\partial t_\lambda}\right)_{t=0} = \sum c_\lambda \rho_0 \left(\frac{\partial}{\partial t}\right) = \theta$

Hence, θ must satisfy the primary obstruction as mentioned in §3.1. These extra properties suggest that m might be smaller than dim $H^1(M, \Theta)$ in general.

We try to count the number of effective parameters for manifolds *already in a complete complex analytic family*:

Definition 4.3 *Let M be a compact complex manifold. If there is an effectively parametrized and complete family $(\mathcal{M}, B, \varpi)$ with $\varpi^{-1}(0) = M, 0 \in B \in \mathbb{C}^m$, define the number of moduli $m(M)$ of M to be $m(M) = \dim B = m$.*

It is necessary to check that $m(M)$ is invariant under the choice of family chosen.

Suppose there exists another effectively parametrized and complete $(\mathcal{N}, D, \pi)$ with $\pi^{-1}(0) = M, 0 \in D \subseteq \mathbb{C}^l$, we need to show that $m = l$.

Since $(\mathcal{N}, D, \pi)$ is complete, there exists Δ with $0 \in \Delta \subseteq B$, a holomorphic $h : \Delta \to D$ such that $h(0) = 0$ and $(\mathcal{M}_\Delta, \Delta, \varpi)$ is induced from $(\mathcal{N}, D, \pi)$ by h, i.e.

$$M_t = \varpi^{-1}(t) = N_{h(t)}, \qquad \forall t \in \Delta$$

Since $t = (t_1, ..., t_m)$ is effective, $\frac{\partial M_t}{\partial t_i} \in H^1(M_t, \Theta_t) = H^1(N_{h(t)}, \Theta_{h(t)}), i = 1, ..., m$ are linearly independent. Meanwhile, $\frac{\partial M_t}{\partial t_i} = \sum_{j=1}^{l} \frac{\partial s_j}{\partial t_i} \frac{\partial N_s}{\partial s_j}$, implying $m \leq l$. Changing the roles of $\mathcal{M}$ and $\mathcal{N}$, we have $m = l$. ■

Theorem 4.3 *Given a compact complex manifold M with $H^2(M, \Theta) = 0$, we have $m(M) = \dim H^1(M, \Theta)$ provided that $m(M)$ is defined.*

PROOF OF THEOREM 4.3

$\because$ $H^2(M, \Theta) = 0$, by the existence theorem (**Theorem 3.2**), we have a family $(\mathcal{M}, B, \varpi)$ with $\varpi^{-1}(0) = M$, $0 \in B \subseteq \mathbb{C}^m$ and $T_0(B) \cong H^1(M, \Theta)$. Then by the completeness theorem (**Theorem 4.1**), $(\mathcal{M}, B, \varpi)$ is complete at 0. From **Definition 4.3**, there exists a complete and effectively parametrized $(\mathcal{N}, D, \pi)$ such that $0 \in D \subseteq \mathbb{C}^{m(M)}$ and $\pi^{-1}(0) = M$.

- $(\mathcal{N}, D, \pi)$ being effectively parametrized implies that $m(M) = \dim T_0(D) \leq \dim H^1(M, \Theta)$.

- The completeness of $(\mathcal{N}, D, \pi)$ implies $\exists \Delta \subseteq B$ containing 0 such that $(\mathcal{M}_\Delta, \Delta, \varpi)$ is induced from $(\mathcal{N}, D, \pi)$ through some $h : \Delta \to D, h(s) = t$ with $h(0) = 0$. Hence, for $\lambda = 1, ..., m$

$$\Big(\frac{\partial M_t}{\partial t_\lambda}\Big)_{t=0} = \Big(\sum_{v=1}^{m(M)} \frac{\partial s_v}{\partial t_\lambda}\frac{\partial N_s}{\partial s_v}\Big)_{s=0} \in H^1(M, \Theta)$$

  are linearly independent. Thus, $m(M) \geq m = \dim T_0(B) = \dim H^1(M, \Theta)$.■

Note that $m(M)$ is not automatically defined. We digress for a moment stating some stability theorems from which some sufficient conditions for the well-definedness of $m(M)$ are derived:

Theorem 4.4 ***(Semi-continuity theorem)*** *Given any differentiable family $(\mathcal{M}, B, \varpi)$ of compact complex manifolds, we have for each $s \in B$, $\exists \varepsilon > 0$ sufficiently small, such that*

$$\dim H^1(M_t, \Theta_t) \leq \dim H^1(M_s, \Theta_s), \qquad |t - s| < \varepsilon.$$

Theorem 4.5 *Suppose* $\dim H^1(M_t, \Theta_t)$ *is t-independent, and $\left(\frac{\partial M_t}{\partial t_k}\right)_{t=0} \in H^1(M_0, \Theta_0), 1 \leq k \leq m$ are linearly independent. Then $\exists \varepsilon > 0$ sufficiently small such that $\frac{\partial M_t}{\partial t_k} \in H^1(M_t, \Theta_t), 1 \leq k \leq m$ are linearly independent for $|t| < \varepsilon$.*

Theorem 4.6 *If* $\dim H^{q-1}(M_t, \Theta_t)$ *and* $\dim H^{q+1}(M_t, \Theta_t)$ *are independent of* $t \in \Delta$*, then so is* $\dim H^q(M_t, \Theta_t)$.

The above 3 theorems will be discussed in **Chapter 6**.

Theorem 4.7 *Suppose* $H^2(M, \Theta) = 0$*, then* $m(M)$ *is defined if and only if* $\dim H^1(M_t, \Theta_t)$ *is t-independent,* $t \in \Delta, 0 \in \Delta \subseteq B$ *is sufficiently small. If the independence condition is satisfied, then* $(\mathcal{M}, \Delta, \varpi)$ *is an effectively parametrized complete family.*

Proof of Theorem 4.7

$\Rightarrow)$ $\because H^2(M, \Theta) = 0$, $\therefore$ By **Theorem 4.3**, $m(M) = \dim H^1(M, \Theta)$

Now, from the argument in **Theorem 4.3**, there exists a complete, effectively parametrized $(\mathcal{N}, D, \pi)$ such that $0 \in D \subseteq \mathbb{C}^{m(M)}, \pi^{-1}(0) = M$.

- D being effective $\Rightarrow \rho_s : T_s(D) \to H^1(N_s, \Theta_s)$ is an injection
  $\Rightarrow \dim H^1(N_s, \Theta_s) \geq \dim T_s(D) = m(M) = \dim H^1(M, \Theta)$

- $(\mathcal{N}, D, \pi)$ being complete $\Rightarrow \exists \Delta \subseteq B$ containing 0 such that $(\mathcal{M}_\Delta, \Delta, \varpi)$ is induced from $(\mathcal{N}, D, \pi)$ through some holomorphic $h : \Delta \to D$ with $h(0) = 0$.
  $\therefore \dim H^1(N_0, \Theta_0) = \dim H^1(M, \Theta) \Rightarrow \dim H^1(N_s, \Theta_s) \geq \dim H^1(N_0, \Theta_0)$.
  This, together with **Theorem 4.4**, leads to $\dim H^1(N_s, \Theta_s) = \dim H^1(N_0, \Theta_0)$, $\forall |s| < \varepsilon, \varepsilon > 0$ sufficiently small.

Thus, shrinking Δ if necessary to allow $|h(t)| < \varepsilon, \forall t \in \Delta$, we have $\dim H^1(M_t, \Theta_t) = \dim H^1(N_s, \Theta_s) = \dim H^1(N_0, \Theta_0) = \dim H^1(M, \Theta) = m(M), \forall t \in \Delta$

$\Leftarrow)$ If $\dim H^1(M_t, \Theta_t) = m$ is independent of t, then $\left(\frac{\partial M_t}{\partial t_\lambda}\right)_{t=0} \in H^1(M_0, \Theta_0), \lambda = 1, ..., m$ are linearly independent. Hence, by **Theorem 4.5**, $\frac{\partial M_t}{\partial t_\lambda} \in H^1(M_t, \Theta_t), \lambda = 1, ..., m$ are linearly independent for all $|t| < \varepsilon, \varepsilon > 0$ sufficiently small.

Let $\Delta_\varepsilon = \{t \in \Delta \| t| < \varepsilon\}$, then $(\mathcal{M}_{\Delta_\varepsilon}, \Delta_\varepsilon, \varpi)$ is effectively parametrized and $\rho_t : T_t(\Delta_\varepsilon) \to H^1(M_t, \Theta_t)$ is surjective for all $t \in \Delta_\varepsilon$. Hence, by the completeness theorem (**Theorem 4.1**), $(\mathcal{M}_{\Delta_\varepsilon}, \Delta_\varepsilon, \varpi)$ is complete. By **Definition 4.3**, $m(M)$ is defined (and is equal to m). ■

Theorem 4.8 *If $H^0(M,\Theta) = H^2(M,\Theta) = 0$, then $m(M)$ is defined and $m(M) = \dim H^1(M,\Theta)$. Moreover, M has an effectively paramtrized complete $(\mathcal{M}_\Delta, \Delta, \varpi)$ with $\varpi^{-1}(0) = M$.*

PROOF OF THEOREM 4.8

The existence theorem (**Theorem 3.2**) guarantees there is $(\mathcal{M}, B, \varpi)$ such that $\varpi^{-1}(0) = M$ and $T_0(B) \cong H^1(M,\Theta)$. Using **Theorem 4.7**, we only need to check that $\dim H^1(M_t, \Theta_t)$ is t-independent for all t small, but this is true by **Theorem 4.5** and **Theorem 4.6** as $H^0(M_t,\Theta_t) = H^2(M_t,\Theta_t) = 0$ for all t small. ■

4.5 Examples

Kodaira and Spencer [6] have fully computed the following examples of complex manifolds, generalizing the already established fact on Riemann surfaces:

M	$m(:= \dim T_0(B))$	$\dim H^1(M,\Theta)$	$m = \dim H^1(M,\Theta)$?
$\mathbb{P}^n$	0	0	Yes
$\mathbb{C}^n/G$, complex tori	n^2	n^2	Yes
Compact Riemann surface of genus $g \geq 2$	$3g-3$	$3g-3$	Yes
Quadratic transform of $\mathbb{P}^2$ (on v distinct points)	$\begin{cases} 0 & \text{if } v \leq 4 \\ 2v-8 & \text{if } v \geq 5 \end{cases}$	$\begin{cases} 0 & \text{if } v \leq 4 \\ 2v-8 & \text{if } v \geq 5 \end{cases}$	Yes
Hypersurface in $\mathbb{P}^{n+1}$ (degree h)	$C_h^{n+1+h} - (n+2)^2$	$\begin{cases} 20 & \text{if } n=2, h=4 \\ C_h^{n+1+h} - (n+2)^2 & \text{otherwise} \end{cases}$	$\begin{cases} \text{NO}^{**} & \text{if } n=2, h=4 \\ \text{Yes} & \text{otherwise} \end{cases}$

****Note:** When $n = 2, h = 4$, we have $C_h^{n+1+h} - (n+2)^2 = 19$.

It should be observed that $m = \dim H^1(M, \Theta)$ in most cases, but not always. Note also that the m's in the above table are NOT necessarily numbers of moduli of the respective manifolds, unless the families chosen can be proved to be effectively parametrized and complete.

Nevertheless, we can compute $m(M)$ by applying the theorems established above:

1. **Projective Space $\mathbb{P}^n$:** We have $m = \dim H^1(M, \Theta) = 0$ since we do not have any nontrivial family for $\mathbb{P}^n$. By the completeness theorem, the family of only one member $\mathbb{P}^n$ is complete, and hence $m(\mathbb{P}^n) = 0$.

2. **Complex Tori:** Since we already have $m = \dim H^1(M, \Theta)$, we only need to check, by the completeness theorem, that the chosen family (whose parameters are the entries of the period matrix) is effectively parametrized, but this is true ([6], p230-4) and hence $m(M) = n^2$.

3. **Hypersurfaces in $\mathbb{P}^{n+1}$:** For cases other than $n = 2, h = 4$, we can make use of the completeness theorem to prove that $M(M) = \dim H^1(M, \Theta)$ as in the case of complex tori. When $n = 2, h = 4$, we need to observe that $H^0(M, \Theta) = H^2(M, \Theta) = 0$, and hence by **Theorem 4.8**, we also have $m(M) = \dim H^1(M, \Theta)$ despite the fact that for our chosen family, $m \neq \dim H^1(M, \Theta)$.

Finally, it should be noted that $m(M)$ is not automatically defined for any given M. Kodaira ([6], p309-14) has given the following example: Consider $M_t = U_1 \times \mathbb{P}^1 \cup U_2 \times \mathbb{P}^1$, where $U_1 = U_2 = \mathbb{C}$, and $(z_1, \zeta_1) \in U_1 \times \mathbb{P}^1$ is glued to $(z_2, \zeta_2) \in U_2 \times \mathbb{P}^1$ through $z_1 z_2 = 1$, $\zeta_1 = z_2^2 \zeta_2 + t z_2$. It can be shown that M_t forms a complete family with $H^2(M_0, \Theta_0) = 0$, but $\dim H^1(M_t, \Theta_t) = 1$ when $t = 0$ and 0 otherwise. Hence, by **Theorem 4.7**, $m(M_0)$ is not defined.

Chapter 5

CR Manifolds and Deformations

In this chapter, we introduce the CR manifolds and its tangential Cauchy Riemann structure as a generalization to complex manifolds. Then the differentiable family of CR manifolds is set up as in **Chapter 2**, and we end this chapter by stating the semi-continuity theorem for the CR case.

5.1 CR Submanifolds and Tangential Complex

Here we state the complexification of vector spaces and complex structures essential for later uses.

1. **Complexification:** Let $V^{(n)}$ be a real vector space, then the complexification of V, denoted as $V \otimes_{\mathbb{R}} \mathbb{C}$ is a complex vector space with (complex) dimension n. Essentially we have

   $$\alpha(v \otimes \beta) = v \otimes \alpha\beta, \quad \overline{v \otimes \alpha} = v \otimes \overline{\alpha}, \forall \alpha, \beta \in \mathbb{C}, \forall v \in V$$

   The dual space $V^* \otimes_{\mathbb{R}} \mathbb{C}$ is defined by the pairing:

   $$< \phi \otimes \alpha, v \otimes \beta > := < \phi, v > \alpha\beta, \ \forall \alpha, \beta \in \mathbb{C}, \forall \phi \in V^*, \forall v \in V.$$

   A real linear map $L : V \to W$ of real vector spaces induces a complex linear map $V \otimes_{\mathbb{R}} \mathbb{C} \to W \otimes_{\mathbb{R}} \mathbb{C}$ by $L(v \otimes \alpha) := (Lv) \otimes \alpha, \forall v \in V, \forall \alpha \in \mathbb{C}$.

2. **Complex structures:** A complex structure map $J : V \to V$ is a real linear map with $J^2 = -\mathrm{id}_V$. By definition, V should be even-dimensional in order to have a complex structure. Again, we can complexify J to give a complex linear map on $V \otimes_{\mathbb{R}} \mathbb{C}$, which is conjugate invariant: $J(\overline{v \otimes \alpha}) = \overline{J(v \otimes \alpha)}$. Thus, we have the decomposition of $V \otimes_{\mathbb{R}} \mathbb{C}$ by the eigenspaces of J:

$$V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1} \quad , \quad \overline{V^{1,0}} = V^{0,1}$$

where $V^{1,0}$ (resp. $V^{0,1}$) is the $+i$ (resp. $-i$) eigenspace of J in $V \otimes \mathbb{C}$.

Lemma 5.1 *Suppose V is an even-dimensional real vector space and $\mathbb{L} \subseteq V \otimes \mathbb{C}$ is a subspace such that $\mathbb{L} \cap \overline{\mathbb{L}} = \{0\}$ and $\mathbb{L} \oplus \overline{\mathbb{L}} = V \otimes \mathbb{C}$, then there exists a unique $J : V \to V$ such that $\mathbb{L}$ and $\overline{\mathbb{L}}$ are $+i$ and $-i$ eigenspace of the complexified map.*

The above treatment can be easily extended to a differentiable manifold.
Now, consider any real submanifold M of $\mathbb{C}^N$. One first looks at

Definition 5.1 *For any $p \in M, H_p(M) := T_p(M) \cap J(T_p(M))$ is called the complex/holomorphic tangent space of M at p. The quotient space $X_p(M) := T_p(M)/H_p(M)$ is called the totally real part of the tangent space at p.*

We shall identify $X_p(M)$ with the orthogonal complement of $H_p(M)$ in $T_p(M)$ relative to the Euclidean inner product. We then have $T_p(M) = H_p(M) \oplus X_p(M)$.

Lemma 5.2 *Suppose M has real codimension d in $\mathbb{C}^N$, (i.e. $\dim_{\mathbb{R}} T(M) = 2N - d$), then $d \leq m \leq 2d, \forall p \in M$, where m is the real codimension of $H_p(M)$ in $T_p(\mathbb{C}^N)$.*

Definition 5.2 *M is called a CR submanifold of CR codimension k in $\mathbb{C}^N$ if the real codimension of $H_p(M)$ in $\mathbf{T_p(M)}$ is constant for all $p \in M$ and is equal to k.*

Note that it is possible for k to vary with $p \in M$.

REMARK

1. **Lemma 5.2** says that $0 \leq k \leq d$. The extreme case $k = 0$ corresponds to a complex manifold while in the other extreme $k = d$, M is called a **generic** CR submanifold.

2. In the case where $H_p(M) = 0$ for all p, M is called totally real. Clearly, a totally real submanifold is of real dimension at most N and is a CR submanifold of CR codimension $k = 2N - d$.

Lemma 5.3 *If M is a real hypersurface in $\mathbb{C}^N$ (i.e. $d = 1$), then M is a CR submanifold of CR codimension $k = 1$.*

PROOF OF LEMMA 5.3

Since J of $\mathbb{C}^N$ restricts to $H_p(M)$, $\dim_{\mathbb{R}} H_p(M)$ is even. **Lemma 5.2** implies that the real codimension of $H_p(M)$ is 2, which in turns implies $k = 1$. ■

Definition 5.3 *Define* $H_p^{1,0}(M) := T_p^{1,0}(\mathbb{C}^N) \cap (T_p(M) \otimes \mathbb{C})$; $H_p^{0,1}(M) := T_p^{0,1}(\mathbb{C}^N) \cap (T_p(M) \otimes \mathbb{C})$

Lemma 5.4 $H_p(M) \otimes \mathbb{C} = H_p^{1,0}(M) \oplus H_p^{0,1}(M)$; $\overline{H_p^{1,0}(M)} = H_p^{0,1}(M)$

PROOF OF LEMMA 5.4

The conjugate relation is obvious by definition. For the other equality, we proceed as follows: Extend $J : H_p(M) \to H_p(M)$ to $J : H_p(M) \otimes \mathbb{C} \to H_p(M) \otimes \mathbb{C}$. This leads to a decomposition of $H_p(M) \otimes (C)$ into $+i$ and $-i$ eigenspaces, say H^+ and H^- respectively. Note that by definition,

$$H_p^{1,0}(M) = \{X \in T_p(M) \otimes \mathbb{C} | J(X) = iX\} \tag{5.1}$$

This implies that

1. $H^+ \subseteq H_p^{1,0}(M)$ since any $+i$ eigenvector of $H_p(M) \otimes \mathbb{C}$ is an eigenvector of $T_p(M) \otimes \mathbb{C}$ ($\because H_p(M) \otimes \mathbb{C} \subseteq T_p(M) \otimes \mathbb{C}$);

2. $H_p^{1,0}(M) \subseteq H^+$ since by (5.1), we also have $H_p^{1,0}(M) \subseteq H_p(M)$.

Similar argument shows that $H^- = H_p^{0,1}(M)$, and we are done. ■

REMARK:

1. Suppose M is a smooth submanifold of $\mathbb{C}^N$ defined by $\{z \in \mathbb{C}^N | \rho_1(z) = ... = \rho_d(z) = 0\}$, with $d\rho_1 \wedge ... \wedge d\rho_d \neq 0$, then it is easy to see for each fixed $p \in M$,

$$H_p^{1,0}(M) = \{W \in T_p^{1,0}(\mathbb{C}^N) | W(\rho_k)(p) = 0, 1 \leq k \leq d\}$$

   (Similar results hold for $H_p^{0,1}(M)$). Hence, $\dim_{\mathbb{C}} H_p^{1,0}(M) = n - k$, where k is the number of linearly independent elements of $\{\partial\rho_1, ..., \partial\rho_d\}$. Note that it may happen that $k < d$. For example, consider $M \subseteq \mathbb{C}^2$ defined by $x_2 = y_2 = 0$, then $d = 2$ but $\dim H_p^{1,0}(M) = 1, \forall p \in M$.

2. For any CR submanifold M of $\mathbb{C}^N$, we have

   - $H_p^{1,0}(M) \cap H_p^{0,1}(M) = \{0\}, \forall p \in M$;
   - $H_p^{1,0}(M)$ and $H_p^{0,1}(M)$ are involutive, i.e. closed under the Lie bracket.

   These properties are important because they are used to define an abstract CR manifold without any ambient space.

From now on, we focus only on any real hypersurface M in $\mathbb{C}^N$, defined by one defining function ρ. Let $\bigwedge^{p,q} T^*(\mathbb{C}^N)$ be the vector bundle of (p,q)-forms on $\mathbb{C}^N$, and $\mathcal{E}^{p,q}(U)$ be the space of smooth sections of $\bigwedge^{p,q} T^*(\mathbb{C}^N)$ over an open set $U \subseteq \mathbb{C}^N$. Then for any hypersurface M defined by ρ in $\mathbb{C}^N$, we have

Definition 5.4 *$\bigwedge^{p,q} T^*(\mathbb{C}^N)|_M$ is the restriction of $\bigwedge^{p,q} T^*(\mathbb{C}^N)$ on M, where a Hermitian inner product is defined as to make $\{dz^I \wedge d\overline{z}^J | \ |I| = p, |J| = q, I, J$ increasing$\}$ an orthonormal basis of $\bigwedge^{p,q} T^*(\mathbb{C}^N)$. In addition, write*

$$I^{p,q} := \{\Phi_1\rho + \Phi_2 \wedge \overline{\partial}\rho \mid \Phi_1 \in \bigwedge^{p,q} T^*(\mathbb{C}^N), \Phi_2 \in \bigwedge^{p,q-1} T^*(\mathbb{C}^N)\},$$

which is an ideal in $\bigwedge^{p,q} T^*(\mathbb{C}^N)$. *The orthogonal complement of* $I^{p,q}|_M$ *in* $\bigwedge^{p,q} T^*(\mathbb{C}^N)|_M$ *is denoted by* $(I^{p,q}|_M)^\perp$.

Let $t_M : \bigwedge^{p,q} T^*(\mathbb{C}^N)|_M \to (I^{p,q}|_M)^\perp$ be the projection map, then the image $t_M(f)$ of $f \in \bigwedge^{p,q} T^*(\mathbb{C}^N)|_M$ is called the **tangential part** of f. Conversely, any smooth section g of $(I^{p,q}|_M)^\perp$ over U, denoted as $g \in \mathcal{E}_M^{p,q}(U)$, can be extended to some $\tilde{g}$ defined on $\tilde{U} \cap \mathbb{C}^N$ with $\tilde{U} \cap M = U$, and we have

Lemma 5.5 *For any* $\tilde{g}_1, \tilde{g}_2 \in \mathcal{E}^{p,q}(\tilde{U})$ *with* $t_M(\tilde{g}_1) = t_M(\tilde{g}_2)$, *we have* $t_M(\overline{\partial}\tilde{g}_1) = t_M(\overline{\partial}\tilde{g}_2)$.

PROOF OF LEMMA 5.5

Given $\rho : \tilde{U} \to \mathbb{R}$ a defining function on $M \cap \tilde{U}$, we have

$$\overline{\partial}(\alpha\rho + \beta \wedge \overline{\partial}\rho) = (\overline{\partial}\alpha)\rho + (\alpha + \overline{\partial}\beta) \wedge \overline{\partial}\rho \in I^{p,q+1},$$

$\forall \alpha \in \mathcal{E}^{p,q}(\tilde{U}), \forall \beta \in \mathcal{E}^{p,q-1}(\tilde{U})$. Hence, $\overline{\partial}(I^{p,q}) \subseteq I^{p,q+1} \Rightarrow t_M\overline{\partial}(I^{p,q}) = 0 \Rightarrow t_M(\overline{\partial}\tilde{g}_1) - t_M(\overline{\partial}\tilde{g}_2) = t_M(\overline{\partial}(\tilde{g}_1 - \tilde{g}_2)) = 0$ since $\tilde{g}_1 - \tilde{g}_2 \in I^{p,q}$ by assumption. ■

By **Lemma 5.5**, we can define the tangential CR complex $\overline{\partial}_M : \mathcal{E}_M^{p,q}(U) \to \mathcal{E}_M^{p,q+1}(U)$ as follows:

$\forall f \in \mathcal{E}_M^{p,q}(U)$, *let* $\tilde{U} \subseteq \mathbb{C}^N$ *be an open set with* $\tilde{U} \cap M = U$, *and let* $\tilde{f} \in \mathcal{E}^{p,q}(\tilde{U})$ *with* $t_M(\tilde{f}) = f$, *then* $\overline{\partial}_M f := t_M(\overline{\partial}\tilde{f})$.

$\overline{\partial}_M$ forms a complex on $\mathcal{E}_M^{p,q}$ much similar to the Cauchy Riemann complex $\overline{\partial}$ on $\mathcal{E}^{p,q}$:

1. $\overline{\partial}_M(f \wedge g) = \overline{\partial}_M f \wedge g + (-1)^{pq} f \wedge \overline{\partial}_M g, \forall f \in \mathcal{E}_M^{p,q}, \forall g \in \mathcal{E}_M^{r,s}$;

2. $\overline{\partial}_M \circ \overline{\partial}_M = 0$.

5.2 Abstract CR Manifolds and its Cohomologies

Definition 5.5 *Let M be a differentiable manifold and S be a subbundle of $T(M) \otimes \mathbb{C}$, then S is called a partially complex structure if $S \cap \overline{S} = 0$, and $[\Gamma(S), \Gamma(S)] \subseteq \Gamma(S)$, where $\Gamma(S)$ is the space of smooth sections of S over M. In this case, we call M a partially complex manifold.*

Obviously this generalizes the concept of complex manifolds, and includes all CR submanifolds as defined in §5.1 as special cases.

Theorem 5.1 *For any partially complex manifold M (with partially complex structure S), there is a unique subbundle P of $T(M)$ such that $P \otimes \mathbb{C} = S \oplus \overline{S}$, and a unique homomorphism $J : P \to P$ such that $J^2 = -id_P$ and $S = \{X - iJX \mid X \in P\}$. ($(P, J)$ is called the real expression of S)*

Proof of Theorem 5.1

It suffices to prove the existence of P such that $P \otimes \mathbb{C} = S \oplus \overline{S}$; Once this is settled, the rest comes from **Lemma 5.1**. But P is defined naturally as the space of all real, linear combinations of basis vectors of the space $S \oplus \overline{S}$. ■

M is called an abstract CR manifold if $\dim T(M)/P = 1$. In what follows, this intrinsic setting of CR manifolds is adopted and the modifier "abstract" is dropped when there is no ambiguity.

For any differentiable manifold M, we define $F(M)$ to be the space of real-valued C^∞ functions on M. Now, consider a CR manifold M with structure S, we define the CR operator $d'' : F(M) \otimes \mathbb{C} \to \Gamma(\overline{S}^*)$ as follows:

$$\forall u \in F(M) \otimes \mathbb{C}, (d''u)(\overline{X}) := \overline{X}u, \forall X \in S_x, x \in M.$$

Definition 5.6 *A complex vector bundle E of M, where M is a CR manifold, is holomorphic if there is some $\overline{\partial}_E : \Gamma(E) \to \Gamma(E \otimes \overline{S}^*)$ such that*

1. $\overline{X}(fu) = \overline{X}f \cdot u + f \cdot \overline{X}u$,
2. $[\overline{X}, \overline{Y}]u = \overline{X}\overline{Y}u - \overline{Y}\overline{X}u$,

$\forall u \in \Gamma(E), \forall f \in F(M) \otimes \mathbb{C}, \forall X, Y \in \Gamma(S)$ *and* $\overline{Z}u = (\overline{\partial}_E u)(\overline{Z}), \forall Z \in \Gamma(S)$.

Remark

1. The trivial line bundle $M \times \mathbb{C}$ is holomorphic with respect to d''.
2. The operator $\overline{\partial}_E$ is called the Cauchy-Riemann operator and a solution u to $\overline{\partial}_E u = 0$ is called a holomorphic cross section.

Definition 5.7 *Let E and F be any CR vector bundles on M, then a bundle homomorphism $\phi : E \to F$ is holomorphic if $\overline{X}(\phi(u)) = \phi(\overline{X}u)$, $\forall u \in \Gamma(E), \forall X \in S$.*

Theorem 5.2 *The factor bundle $T^{1,0}(M) := T(M) \otimes \mathbb{C}/\overline{S}$ of a CR manifold (M, S) is a holomorphic vector bundle with respect to $\overline{\partial} = \overline{\partial}_{T^{1,0}(M)} : \Gamma(T^{1,0}(M)) \to \Gamma(T^{1,0}(M) \otimes \overline{S}^*)$ as defined below: $\forall u \in \Gamma(T^{1,0}(M))$, choose $Z \in \Gamma(T(M) \otimes \mathbb{C})$ such that $u = \omega(Z)$, where $\omega : T(M) \otimes \mathbb{C} \to T^{1,0}(M)$ is the projection, and write $(\overline{\partial}u)(\overline{X}) = \omega([\overline{X}, Z]), \forall X \in \Gamma(S)$*

Proof of Theorem 5.2

We need to check that $\overline{\partial}$ is well-defined, and satisfies the two conditions in **Definition 5.6**

Suppose we have $Z_1, Z_2 \in \Gamma(T(M) \otimes \mathbb{C})$ such that $u = \omega(Z_1) = \omega(Z_2)$, then $Z_1 - Z_2 \in \overline{S} \Rightarrow \overline{Z_1 - Z_2} \in S$. Hence, $\forall X \in \Gamma(S)$,

$$[\overline{X}, Z_1] - [\overline{X}, Z_2] = [\overline{X}, Z_1 - Z_2] = \overline{[X, \overline{Z_1 - Z_2}]} \in \overline{S},$$

since S is involutive. Thus, $\omega([\overline{X}, Z_1]) = \omega([\overline{X}, Z_2])$ and $\overline{\partial}$ is well-defined. Now, for the two conditions, consider any $Z \in \Gamma(T(M) \otimes \mathbb{C})$ such that $u = \omega(Z)$, and we have:

$$\begin{aligned}
1. \quad \overline{X}(fu) &= (\overline{\partial}(fu))(\overline{X}) = \omega([\overline{X}, fZ]) = \omega(\overline{X}f \cdot Z + f[\overline{X}, Z]) \\
&= (\overline{X}f)\omega(Z) + f \cdot \omega([\overline{X}, Z]) = \overline{X}f \cdot u + f \cdot \overline{X}u \\
2. \quad [\overline{X}, \overline{Y}]u &= (\overline{\partial}u)([\overline{X}, \overline{Y}]) = \omega([[\overline{X}, \overline{Y}], Z]) = -\omega[[\overline{Y}, Z], \overline{X}] - \omega[[Z, \overline{X}], \overline{Y}] \\
&= \omega[\overline{X}, [\overline{Y}, Z]] - \omega[\overline{Y}, [\overline{X}, Z]] \\
&= \overline{\partial}(\omega[\overline{Y}, Z])(\overline{X}) - \overline{\partial}(\omega[\overline{X}, Z])(\overline{Y}) \\
&= \overline{\partial}((\overline{\partial}u)(\overline{Y})(\overline{X}) - \overline{\partial}((\overline{\partial}u)(\overline{X}))(\overline{Y}) \\
&= \overline{\partial}(\overline{Y}u)(\overline{X}) - \overline{\partial}(\overline{X}u)(\overline{Y}) = \overline{X}(\overline{Y}u) - \overline{Y}(\overline{X}u). \blacksquare
\end{aligned}$$

REMARK Consider the case where M is a CR hypersurface of $\mathbb{C}^N$, then $T^{1,0}(M)$ is the holomorphic vector bundle of tangent vectors of type (1,0) to M as defined in §5.1 (Hence, this justifies using the same symbol $T^{1,0}(M)$ for both cases). Indeed, the inclusion $T(M) \otimes \mathbb{C} \to T(\mathbb{C}^N) \otimes \mathbb{C}$ induces an injective homomorphism $T(M) \otimes \mathbb{C}/\overline{S} \to T(\mathbb{C}^N) \otimes \mathbb{C}/(T^{0,1}(\mathbb{C}^N)|_M) = T^{1,0}(\mathbb{C}^N)|_M$. Hence, when $\dim_{\mathbb{C}} T(M) \otimes \mathbb{C}/\overline{S} = N$ (e.g. when M is a hypersurface), we can identify $T(M) \otimes \mathbb{C}/\overline{S}$ with $T^{1,0}(\mathbb{C}^N)|_M$.

Let E be a holomorphic vector bundle over E, and write for all $q \geq 0$,

$$C^q(M, E) = E \otimes \bigwedge^q \overline{S}^*; \qquad \mathscr{C}^q(M, E) = \Gamma(C^q(M, E))$$

and $\overline{\partial}_E^q : \mathscr{C}^q(M, E) \to \mathscr{C}^{q+1}(M, E)$ defined by

$$\begin{aligned}
(\overline{\partial}_E^q \phi)(\overline{X}_1, ..., \overline{X}_q) = \sum_{i=1}^{q+1} (-1)^{i+1} \overline{X}_i (\phi(\overline{X}_1, ..., \widehat{\overline{X}}_i, ..., \overline{X}_q)) \\
\sum_{i<j} (-1)^{i+j} \phi([\overline{X}_i, \overline{X}_j], \overline{X}_1, ..., \widehat{\overline{X}}_i, ..., \widehat{\overline{X}}_j, ..., \overline{X}_q),
\end{aligned}$$

for all $\phi \in \mathscr{C}^q(M, E), X_1, ..., X_q \in \Gamma(S)$. Since we also have $\overline{\partial}_E^{q+1} \circ \overline{\partial}_E^q = 0$, $\{\mathscr{C}^q(M, E), \overline{\partial}_E^q\}$ is a complex giving rise to cohomology groups, denoted as

$H^q(M, E)$.

We digress for a moment to define the spectral sequence $\{E_r^{p,q}(M)\}$ for any de Rham complex $\{\mathscr{A}^k(M), d\}$ of a differentiable manifold M. Denote

$$\begin{aligned} A^k(M) &:= \bigwedge^k (T(M)\otimes\mathbb{C})^* \qquad\qquad \mathscr{A}^k := \Gamma(A^k(M)) \\ F^p(A^k(M)) &:= \{\phi\in A^k(M) \mid \phi(X_1,...,X_{p-1},\overline{Y}_1,...,\overline{Y}_{k-p+1}) = 0, \\ &\qquad\qquad \forall X_1,...,X_{p-1}\in T_x(M)\otimes\mathbb{C}, Y_1,...,Y_{k-p+1}\in S_x\} \\ F^p(\mathscr{A}^k(M)) &= \Gamma(F^p(A^k(M))) \end{aligned}$$

Thus, we have

$$\begin{aligned} &F^p(A^k(M)) \supseteq F^{p+1}(A^k(M)), \qquad && dF^p(\mathscr{A}^k(M)) \subseteq F^p(\mathscr{A}^{k+1}(M)) \\ &F^0(A^k(M)) = A^k(M), && F^{p+1}(A^p(M)) = 0 \end{aligned}$$

We now define $H^{p,q}(M)$ as follows:

$$\begin{aligned} &A^{p,q}(M) = F^p(A^{p+q}(M)), \qquad && \mathscr{A}^{p,q}(M) = \Gamma(A^{p,q}(M) \\ &C^{p,q}(M) = A^{p,q}(M)/A^{p+1,q+1}(M), && \mathscr{C}^{p,q}(M) = \Gamma(C^{p,q}(M)) \end{aligned}$$

Then $H^{p,q}(M)$ are the cohomology groups of $\{\mathscr{C}^{p,q}(M), d''\}$, where $d'' : \mathscr{C}^{p,q}(M) \to \mathscr{C}^{p,q+1}(M)$ is induced naturally from $d : \mathscr{A}^{p,q}(M) \to \mathscr{A}^{p,q+1}(M)$.

Theorem 5.3 *$C^{p,q}(M)$ may be identified with $C^q(M, E^p)$ and $\forall\phi\in\mathscr{C}^{p,q}(M), d''\phi = (-1)^p\overline{\partial}_{E^p}\phi$, where $E^p = \bigwedge^p(T^{0,1}(M))^*$ is a CR vector bundle by*

$$(\overline{Y}\psi)(u_1,...,u_p) = \overline{Y}(\psi(u_1,...,u_p)) + \sum_i (-1)^i\psi(\overline{Y}u_i, u_1,...,\widehat{u}_i,...,u_p),$$

$\forall\psi\in\Gamma(E^p), u_1,...,u_p\in\Gamma(T^{0,1}(M)), Y\in S$.

Proof of Theorem 5.3

We have the following short exact sequence:

$$0 \longrightarrow A^{p+1,q-1}(M) \hookrightarrow A^{p,q}(M) \xrightarrow{\iota^p} C^q(M, E^p) \longrightarrow 0$$

where $\iota^p : A^{p,q}(M) \to C^q(M, E^p)$ is defined by

$$(\iota^p\phi)(\omega(X_1), ..., \omega(X_p), \overline{Y}_1, ..., \overline{Y}_q) = \phi(X_1, ..., X_p, \overline{Y}_1, ..., \overline{Y}_q),$$

$\forall \phi \in A^{p,q}(M), \forall X_1, ..., X_p \in T_x(M) \otimes \mathbb{C}, \forall Y_1, ..., Y_q \in S_x$. The mapping is well-defined by the definition of $A^{p,q}$. Hence, $C^{p,q}(M) = A^{p,q}(M)/A^{p+1,q-1}(M) \cong C^q(M, E^p)$. Moreover, $\overline{\partial}_{E^p}\iota^p\phi = (-1)^p\iota^p d\phi = (-1)^p d''(\iota^p\phi), \forall \phi \in \mathscr{A}^{p,q}(M)$. ■

REMARK When M is a complex manifold, then $H^{p,q}(M)$ are exactly the Dolbeault cohomology groups, and $H^q(M, E) \cong H^q(M, \Omega^0(E))$, where $\Omega^0(E)$ is the sheaf of germs of local holomorphic sections of E.

5.3 Strongly Pseudoconvex Manifolds

Let M be a differentiable manifold and P a subbundle of $T(M)$. Denote $P' = T(M)/P$ and write $\varpi : T(M) \to P'$ as the canonical projection. We can define $\omega \in \Gamma(P' \otimes \bigwedge^2 P^*)$ by

$$\omega(X, Y) := \varpi([X, Y]), \qquad \forall X, Y \in \Gamma(P)$$

If (M, S) is a CR manifold and (P, J) is a real expression of S, then $\forall X, Y \in \Gamma(P)$, we have

$$\begin{aligned}
&X - iJX, Y - iJY \in S \qquad\qquad (\because (P, J) \text{ is a real expression of } S)\\
&\Rightarrow [X - iJX, Y - iJY] \in S\\
&\Rightarrow [X, Y] - [JX, JY] - iJ([X, Y] - [JX, JY]) \in S\\
&\Rightarrow [JX, JY] - [X, Y] \in \Gamma(P)\\
&\therefore \omega(JX, JY) = \varpi([JX, JY]) = \varpi([X, Y]) = \omega(X, Y)
\end{aligned}$$

This defines a P'_x-valued Hermitian form L_x on P_x for each $x \in M$,

$$L_x(X, Y) := \omega(JX, Y), \qquad \forall X, Y \in \Gamma(P)$$

Definition 5.8 *(M, S) is a strongly pseudoconvex (s.p.c.) manifold if (M, S) is a CR manifold and L_x is definite at each $x \in M$, i.e. when $L_x(X, X) = 0, X \in P_x$, then $X = 0$.*

Theorem 5.4 *If (M, S) is a s.p.c. manifold, then*

1. *If $X \in P_x, \omega(X, Y) = 0, \forall Y \in P_x$, then $X = 0$. (In this case ω_x is said to be nondegenerate at each $x \in M$.)*

2. *P is trivial and there is a trivialization θ of $(P')^*$ (i.e. $\theta \in \Gamma(P')^*$ such that $\theta_x \neq 0, \forall x \in M$) such that*

   (a) *$d\theta(X, Y) = -\theta([X, Y]), \forall X, Y \in \Gamma(P)$ and hence $d\theta_x$ restricted to P is nondegenerate at each $x \in M$.*

   (b) *There is a unique* ***infinitesimal contact transformation*** *ξ on P (i.e. $[\xi, \Gamma(P)] \subseteq \Gamma(P)$) such that $\theta(\xi) = 1, \xi \lrcorner d\theta = 0$.*

   (c) *$L_x(X, Y) = -d\theta(JX, Y) \cdot \varpi(\xi_x)$.*

   (d) *The Hermitian form $-d\theta(JX, X), X \in P_x$ is positive definite at each $x \in M$.*

REMARK

1. A s.p.c. manifold must be odd dimensional.

2. The mapping $\theta \to \xi$ gives a 1-1 correspondence between the trivializations on $\Gamma(P^*)$ and the infinitesimal contact transformation ξ such that $\xi_x \notin P_x, \forall x \in M$.

Definition 5.9 *A trivialization θ of $(P')^*$ is called a* ***basic form*** *if it satisfies 2d of* ***Theorem 5.4****. An infinitesimal contact transformation ξ is called a* ***basic field*** *if $\xi_x \notin P, \forall x \in M$ and the corresponding θ is a basic form.*

It follows that a 1-form θ' is a basic form if and only if $\theta' = f\theta$ for some positive function f and some basic form θ. In addition, any s.p.c. manifold can be given an orientation $\theta \wedge (d\theta)^{n-1}$, where θ is a basic form.

We now justify the usage of "strongly pseudoconvex manifolds" as follows: Let (M', S') be an N-dimensional complex manifold (or more simply $M' = \mathbb{C}^N$), and $\rho : U \to \mathbb{R}$, U is an open set of M' such that $d\rho_x \neq 0, \forall x \in U$. Define $S(\rho)_x := \{X \in S'_x \mid d\rho(X) = 0\}$, and a Hermitian form $L(f)_x : S(\rho)_x \times S(\rho)_x \to \mathbb{R}$ by

$$L(f)_x(X,Y) := (d'd''\rho)(X,\overline{Y}), \qquad \forall X, Y \in S(\rho)_x$$

Consider the real hypersurface $M = \rho^{-1}(0)$, assuming the set is nonempty. Let S be the CR structure of M with real expression (P, J), then we have $S_x = S(\rho)_x, \forall x \in M$.

Define a 1-form θ on M by $\theta = \sqrt{-1}\iota^* d''\rho$, where $\iota : M \to M'$ is the inclusion, then

1. $\theta = -\sqrt{-1}\iota^* d'\rho$ (and hence θ is real). This is because $d\rho = 0$ on $T(M) \Rightarrow d'\rho = -d''\rho$ there.

2. θ gives a trivialization of $(P')^*$ and $L(f)_x(X,Y) = -\sqrt{-1}d\theta(X,\overline{Y}), \forall X, Y \in S_x$.

Hence, M is s.p.c. as a CR manifold if and only if M is s.p.c. in M' in the classical sense.

5.4 Differentiable Family

Finally, we shall introduce the differentiable family in the CR case:

Definition 5.10 *Let Ω be a domain in $\mathbb{R}^l$ and $\{M_t \mid t \in \Omega\}$ be a family of compact strongly pseudoconvex CR manifolds parametrized by Ω. Then $\{M_t \mid t \in \Omega\}$ is differentiable if there is a fibered manifold $\mathcal{M}$ over Ω with projection π such that*

1. *π is proper;*
2. *$\forall t \in \Omega, \pi^{-1}(t) = M_t$ is a differentiable manifold;*
3. *Let S_t be the strongly pseudoconvex CR structure of M_t, where $S_t \subseteq \mathbb{C}TM_t \subseteq \mathbb{C}T\mathcal{M}$, then $\cup_{t\in\Omega} S_t$ forms a differentiable subbundle of $\mathbb{C}T\mathcal{M}$.*

Similar to the case of complex manifolds, we also have the following semi-continuity theorem (ref: **Theorem 4.4**):

Theorem 5.5 *(**Tanaka**, [15]) Let $\{M_t \mid t \in \Omega\}$ be a differentiable family of compact s.p.c. CR manifolds of dimension $2n-1 \geq 5$, then for $q \neq 0, n-1$, $\dim H^{p,q}(M_t)$ is upper semi-continuous in $t \in \Omega$.*

We shall prove this theorem in the last chapter.

Chapter 6

Stability Theorems

A stability theorem asserts that a certain property of a complex analytic manifold is stable under "sufficiently small deformations" of the complex analytic structure - more precisely that, if a certain fiber V_0 of a differentiable family $\varpi : \mathcal{V} \to M$ possesses the property, then all neighbouring fibres also possess the property. (Kodaira-Spencer [8], p.351)

In this chapter, we first prove the (upper) semi-continuity theorem for compact complex manifolds, and then outline the proof for the case of s.p.c. CR manifolds. Proofs of theorems as stated in previous chapters (**Theorem 2.3**, **Lemma 2.2**, **Theorem 4.5** and **4.6**) are outlined, if not detailed. To accomplish all these proofs, one needs to study the differentiable family of differential operators, and make use of some elliptic and subelliptic estimates to be appeared in the following sections.

6.1 Semi-continuity Theorem

6.1.1 The Case of Compact Complex Manifolds (Theorem 4.4)

The general setting in this section is as follows: Consider a family $\{B_t \mid t \in \Delta\}$ of holomorphic vector bundles over a compact differentiable manifold X. The collection of C^∞ sections of B_t is denoted by $L(B_t)$ and a differential operator is a map $E_t : L(B_t) \to L(B_t)$ which can be locally expressed as

$$(E_t\psi_t)_i^\lambda(x) = \sum_v E_{iv}^\lambda(x,t,D_i)\psi_i^v(x), \qquad \forall \psi_t(x) = (\psi_{ti}^\lambda(x)) \in L(B_t), \forall x \in X$$

where $E_{iv}^\lambda(x,t,D_i)$ is a polynomial in $D_i = \frac{\partial}{\partial x_i^\alpha}$.

Now, given a family of compact complex manifolds $\{M_t \mid t \in \Delta\}$, we first make use of **Theorem 2.6** to assert that M_t is diffeomorphic to the same manifold, say X. Hence, we can treat $B_t = T(M_t) \otimes \wedge^p T^*(M_t) \wedge^q \overline{T}^*(M_t)$ as a differentiable family of holomorphic vector bundles over X, and this family is locally trivial by using arguments as in the proof of **Theorem 2.4**. From now on, write the corresponding $L(B_t)$ as $\mathscr{L}^{p,q}(M_t)$. The Hermitian metric $< \cdot, \cdot >_t$ on $T(M_t) \times \wedge^p T^* M_t \wedge^q \overline{T}^*(M_t)$ induces an inner product on $\mathscr{L}^{p,q}(M_t)$ by defining for all $\psi_t, \phi_t \in \mathscr{L}^{p,q}(M_t)$,

$$(\psi_t, \phi_t)_t := \int_{M_t} < \psi_t, \phi_t >_t \frac{\omega_t^n}{n!},$$

where ω_t^n is the volume form induced by the Hermitian form on M_t. The formal adjoint ϑ_t and the Laplacian $\Box_t := \overline{\partial}_t \vartheta_t + \vartheta_t \overline{\partial}_t$ are defined with respect to $(\cdot,\cdot)_t$ as detailed in **Appendix A**.

Consider any locally finite open covering $\{\mathscr{U}_j\}$ of X, then by the C^∞ equivalence of $\{M_t \mid t \in \Delta\}$ with $X \times \Delta$, we have some C^∞ functions $z_j(x,t)$ on $\mathscr{U}_j$

such that for each $t \in \Delta$, $\{x \mapsto (z_j^1(x,t), ..., z_j^n(x,t)) \mid \mathscr{U}_j \cap X \times t \neq \emptyset\}$ forms a system of local coordinates on M_t. Then for any $\phi_t \in \mathscr{L}^{p,q}(M_t)$, ϕ_t can be represented on each $U_j \times t \subseteq M_t$ (U_j's cover X and $\overline{U_j \times \Delta} \subseteq \mathscr{U}_j$) by

$$\phi_t = (\phi_j^1(z_j,t), ..., \phi_j^v(z_j,t)), \qquad z_j = z_j(x,t),$$

where $\phi_j^\lambda(z_j,t) = \sum \phi^\lambda_{jA_p\overline{B}_q}(z_j,t) dz_j^{A_p} \wedge \overline{dz_j^{B_q}}$, and $\dim T(M_t) = v$.

Definition 6.1 *$\phi_t \in \mathscr{L}^{p,q}(M_t)$ is C^∞ differentiable in $t \in \Delta$ if each $\phi_j^\lambda(z_j,t)$, $\lambda = 1, ..., v$ are C^∞ in z_j and t.*

We define the Sobolev k-norm for $k = 0, 1, 2, ...$ on $\mathscr{L}^{p,q}(M_t)$ as follows:

Definition 6.2 *Fix a locally finite covering U_j on X. Denote any arbitrary differential operator of rank l by D_j^l, then for all $\psi_t \in \mathscr{L}^{p,q}(M_t)$, define*

$$||\psi_t||_k = \Big(\sum_{l=0}^{k} \sum_j \sum_{D_j^l} \int_{U_j} \sum_\lambda |D_j^l \psi_{tj}^\lambda(x)|^2 dX_j \Big)^{\frac{1}{2}}$$

REMARK $||\psi_t||_0$ is equivalent to $||\psi_t||_t$ uniformly in t.

Now we state two important estimates for later use:

1. **(Sobolev's inequality)** For all $l = 0, 1, 2, ...$, any $k \in \mathbb{N}, k > \dim M_t/2, \exists c_{k,l}$ such that

$$|D_j^l \psi_{tj}^\lambda(x)| \leq c_{k,l} ||\psi_t||_{k+l}, \qquad \forall \psi_t \in \mathscr{L}^{p,q}(M_t) \tag{6.1}$$

2. **(Friedrich's inequality)** For all $l = 0, 1, 2, ..., \exists c_k$ independent of t such that

$$(||\psi_t||_{k+m})^2 \leq c_k (||\Box_t \psi_t||_k^2 + ||\psi_t||_0^2), \qquad \forall \psi_t \in \mathscr{L}^{p,q}(M_t) \tag{6.2}$$

The Laplacian $\Box_t$ is self-adjoint with respect to $(\cdot,\cdot)_t$. Let $\mathbf{H}^{p,q}(M_t) = \{\psi_t \in \mathscr{L}^{p,q}(M_t) \mid \Box_t \psi_t = 0\}$, then we have the following orthogonal decompositions:

$$\begin{aligned} \mathscr{L}^{p,q}(M_t) &= \mathbf{H}^{p,q}(M_t) \oplus \Box_t \mathscr{L}^{p,q}(M_t) \\ &= \mathbf{H}^{p,q}(M_t) \oplus \text{ Im } \overline{\partial}_t \oplus \text{ Im } \vartheta_t \end{aligned}$$

Furthermore, from the analysis of compact operators, we have

Theorem 6.1 *$\Box_t$ has real eigenvalues $0 \leq \lambda_1(t) \leq \lambda_2(t) \leq ...$ with $\lim_{h\to\infty} \lambda_h(t) = +\infty$. Let $\psi_{t1}, \psi_{t2}, ...$ be the corresponding orthonormal eigenvectors, then $\forall \phi_t \in \mathscr{L}^{p,q}(M_t)$, $\phi_t = \sum_h (\phi_t, \psi_{th})_t \psi_{th}$, with convergence with respect to $||\cdot||_t$.*

Hence we can define the orthogonal projection $F_t : \mathscr{L}^{p,q}(M_t) \to \mathbf{H}^{p,q}(M_t)$ with

$$F_t\phi_t := \sum_{\lambda_h(t)=0} (\phi_t, \psi_{th})_t \psi_{th}, \qquad \forall \phi_t \in \mathscr{L}^{p,q}(M_t)$$

The Green operator is then defined as $G_t : \mathscr{L}^{p,q}(M_t) \to \text{ Im } \Box_t$ by $G_t\phi_t := \sum_{\lambda_h(t)\neq 0} \frac{(\phi_t, \psi_{th})_t}{\lambda_h(t)} \psi_{th}$. In other words, we have $\Box_t G_t = G_t \Box_t = 1 - F_t$.

Lemma 6.1 *$\sum b_{th}\psi_{th}$ defines an element in $\mathscr{L}^{p,q}(M_t)$ if and only if $\sum_{h=1}^{\infty} \lambda_h(t)^{2l} |b_{th}|^2$ converges for all $l = 0, 1, 2, ...$*

Proof of Lemma 6.1

$\Rightarrow$) Suppose $\sum b_{th}\psi_{th} = \phi_t \in \mathscr{L}^{p,q}(M_t)$, then for all $l = 0, 1, 2, ...$, $\Box_t^l \phi_t = \sum \lambda_h(t)^l b_{th} \psi_{th}$, and $\sum \lambda_h(t)^{2l} |b_{th}|^2$ converges to $||\Box_t^l \phi_t||_t^2$.

$\Leftarrow$) Suppose $\sum_{h=1}^{\infty} \lambda_h(t)^{2l} |b_{th}|^2 < +\infty, l = 0, 1, 2, ...$, then since $\lim_{h\to\infty} \lambda_h(t) = +\infty$, we have

$$\sum_{h=1}^{\infty} \Big(1 + \sum_{l=1}^{k} \lambda_h(t)^{2l}\Big) |b_{th}|^2 < +\infty, \quad k = 1, 2, ... \tag{6.3}$$

By inductively using (6.2), we should also notice that $\forall \psi_t = \sum_{h=1}^{\infty} a_{th}\psi_{th} \in \mathscr{L}^{p,q}(M_t)$,

$$||\psi_t||_{qm}^2 \leq c_m \Big(||\psi_t||_0^2 + \sum_{l=1}^{q} ||\Box_t^l \psi_t||_0^2\Big), \qquad \forall q, m \in \mathbb{N},$$

where c_m is independent of t. For any $k \in \mathbb{N}$,choose q so that $qm - m < k \leq qm$, then

$$\begin{aligned}
||\psi_t||_k^2 \leq ||\psi_t||_{qm}^2 &\leq c_m\Big(||\psi_t||_0^2 + \sum_{l=1}^{q} ||\Box_t^l \psi_t||_0^2\Big) \\
&\leq \hat{c}_m\Big(||\psi_t||_t^2 + \sum_{l=1}^{q} ||\Box_t^l \psi_t||_t^2\Big) \\
\Rightarrow ||\psi_t||_k^2 &\leq \hat{c}_m \sum_{h=1}^{\infty}\Big(1 + \sum_{l=1}^{k} |\lambda_h(t)|^{2l}\Big)|b_{th}|^2 \qquad (6.4)
\end{aligned}$$

for some constant $\hat{c}_m$ since $||\cdot||_0$ and $||\cdot||_t$ are equivalent. Now, consider $\psi_t^{(q)} = \sum_{h=1}^{q} b_{th}\psi_{th}$, then $\psi_t^{(q)} \in \mathscr{L}^{p,q}(M_t)$. To show that $\psi_t = \sum_{h=1}^{\infty} \lambda_h(t)\psi_{th}$ is also in $\mathscr{L}^{p,q}(M_t)$,it suffices to show that for any differential operator D_k^l of order l, the derivatives $D_k^l \psi_j^{(q)\lambda}(x)$ converges locally uniformly. But this is so because for each l, we can choose k large so that $k > l + n/2$, then

$$\begin{aligned}
|D_j^l \psi_j^{(q)\lambda}(x) - D_j^l \psi_j^{(p)\lambda}(x)|^2 &\leq (c_{k-l,l})^2 ||\psi^{(q)} - \psi^{(p)}||_k^2 && (\because (6.1)) \\
&\leq (c_{k-l,l})^2 \hat{c}_m \sum_{h=p+1}^{q}\Big(1 + \sum_{\alpha=1}^{k} |\lambda_h(t)|^{2\alpha}\Big)|b_{th}|^2 && (\because (6.4)),
\end{aligned}$$

which converges to 0 as $p, q \to \infty$, due to (6.3). ■

Corollary 6.1 *Define* $W = \{(t,\zeta) \in \Delta \times \mathbb{C} \mid \zeta \neq \lambda_h(t), \forall h\}$, *then* $\forall (t,\zeta) \in W$, $\Box_t(\zeta) := \Box_t - \zeta$ *is bijective on* $\mathscr{L}^{p,q}(M_t)$.

PROOF OF COROLLARY 6.1

1. Injectivity: If $\Box_t(\zeta)\phi_t = 0$, then $\Box_t\phi_t = \zeta\phi_t \Rightarrow \phi_t = 0$, since $\zeta \neq \lambda_h(t)$ for all h.

2. Surjectivity: Expand any $\phi_t \in \mathscr{L}^{p,q}(M_t)$ as $\phi_t = \sum b_{th}\psi_{th}$. It suffices to show that the formal series $\psi_t = \sum \frac{b_{th}}{\lambda_h(t)-\zeta}\psi_{th}$ is in $\mathscr{L}^{p,q}(M_t)$, and $\Box_t(\zeta)\psi_t = \phi_t$, but these are direct applications of **Lemma 6.1**. ■

Lemma 6.2 W *is an open subset of* $\Delta \times \mathbb{C}$.

PROOF OF LEMMA 6.2

Consider any $(t_q, \zeta_q) \in (\Delta \times \mathbb{C})\backslash W, q = 1, 2, 3, ...$ with $\lim_{q\to\infty}(t_q, \zeta_q) = (t_0, \zeta_0) \in \Delta \times \mathbb{C}$. It suffices to prove that $(t_0, \zeta_0) \notin W$.

Since $(t_q, \zeta_q) \notin W, \exists \phi_{t_q} \in \mathscr{L}^{p,q}(M_{t_q})$ such that $||\phi_{t_q}||_{t_q} = 1, \Box_{t_q}\phi_{t_q} = \zeta_q\phi_{t_q}$. Consider any local chart (U_k, h_k) and any $l \geq 0$, then as in the proof of **Lemma 6.1**, if $m \gg l$, then

$$|D_k^{l+1}\phi^k_{tqI\bar{J}}(x)|^2 \leq c_1||\phi_{t_q}||^2_{(m)} \leq c_2(1 + \zeta_q^2 + ... + \zeta_q^{2m}) < +\infty,$$

$\forall q \geq 0$, for all $x \in K$, an arbitrary compact subset on U_k.

$\therefore$ $D_k^l\phi^k_{tqI\bar{J}}$ are equicontinuous on K, and by passing to subsequences, we may assume $D_k^l\phi^k_{tqI\bar{J}}$ converges locally uniformly for all $l \leq 2$. This implies that $\phi_{t_0} := \lim_{q\to\infty}\phi_{t_q}$ is a (p, q)-vector form with C^2 coefficients, with $||\phi_{t_0}||_{t_0} = 1$. As $\Box_t$ is of order 2, with C^∞ coefficients in (x, t), we have

$$\Box_{t_0}\phi_{t_0} = \lim_{q\to\infty}\Box_{t_q}\phi_{t_q} = \lim_{q\to\infty}\zeta_q\phi_{t_q} = \zeta_0\phi_{t_0} \Rightarrow (t_0, \zeta_0) \notin W. \blacksquare$$

Corollary 6.2 $\forall(t_0, \zeta_0) \in W, \exists\delta > 0, c > 0$ *such that* $\forall|t - t_0| < \delta, |\zeta - \zeta_0| < \delta$ *and* $\phi_t \in \mathscr{L}^{p,q}(M_t), ||\Box_t(\zeta)\phi_t||_0 \geq c||\phi_t||_0$.

PROOF OF COROLLARY 6.2

By **Lemma 6.2**, $\forall(t_0, \zeta_0) \in W, \exists\delta > 0$ such that $\forall\ |t - t_0|, |\zeta - \zeta_0| < \delta, (t, \zeta) \in W \Rightarrow\ |\lambda_h(t) - \zeta| > \delta$ for all $h \Rightarrow \forall\phi_t \in \mathscr{L}^{p,q}(M_t)$, write $\phi_t = \sum a_{th}\psi_{th}$ and get

$$\begin{aligned}||\Box_t(\zeta)\phi_t||_t^2 &= ||\sum(\lambda_h(t) - \zeta)a_{th}\psi_{th}||_t^2 = \sum|\lambda_h(t) - \zeta|^2|a_{th}|^2 \geq \delta^2\sum|a_{th}|^2 \\ &= \delta^2||\phi_t||_t^2\end{aligned}$$

Then the equivalence of $||\cdot||_t$ and $||\cdot||_0$ (uniform in t) gives the desired result. $\blacksquare$

We need the following theorem to establish the differentiability of differential operators on $\mathscr{L}^{p,q}(M_t)$:

Theorem 6.2 *Suppose $E_t : \mathscr{L}^{p,q}(M_t) \to \mathscr{L}^{p,q}(M_t)$ is a differentiable family of linear differentiable operators, bijective in each $t \in \Delta$, and satisfies*

$$||\phi_t||_{m+1}^2 \leq c'_m(||E_t\phi_t||_m^2 + ||\phi_t||_0^2), \qquad ||E_t\phi_t||_0 \geq c||\phi_t||_0$$

uniformly in t and ϕ_t, then ϕ_t being C^∞ in t implies $\psi_t := E_t^{-1}\phi_t$ is also C^∞ in t. We say that E_t^{-1} is $\mathbf{C^\infty}$ ***differentiable in*** $\mathbf{t \in \Delta}$.

The proof of this theorem is deferred to **Appendix C**. With this theorem at hand and applying **Corollary 6.2**, we can conclude that $G_t(\zeta) := \Box_t(\zeta)^{-1}$ is C^∞ differentiable in $(t, \zeta) \in W$ (treating ζ as one of the parameters). Now, fix $t_0 \in \Delta$ and choose a closed Jordan curve C with interior C^0 in $\mathbb{C}$ not passing through any $\lambda_h(t_0)$. Choose $\delta > 0$ as guaranteed by **Lemma 6.2** so that $C \times [t_0 - \delta, t_0 + \delta] \subseteq W$. Define $\forall\phi \in \mathscr{L}^{p,q}(M_t), \forall|t - t_0| < \delta$,

$$F_t(C)\phi_t = \sum_{\lambda_h(t)\in C^0} (\phi_t, \psi_{th})\psi_{th}$$

Theorem 6.3 *$F_t(C)$ is C^∞ differentiable in $t \in \Delta$.*

Proof of Theorem 6.3

The trick is to prove that $F_t(C)\phi_t = -\frac{1}{2\pi i}\int_C G_t(\zeta)\phi_t d\zeta$, using Cauchy integral formula: $\forall\phi_t = \sum b_{th}\psi_{th} \in \mathscr{L}^{p,q}(M_t)$,

$$\begin{aligned} F_t(C)\phi_t = \sum_{\lambda_h(t)\in C^0} b_{th}\psi_{th} &= -\frac{1}{2\pi i}\int_C \sum \frac{b_{th}}{\lambda_h(t) - \zeta}\psi_{th} d\zeta \\ &= -\frac{1}{2\pi i}\int_C G_t(\zeta)\phi_t d\zeta. \blacksquare \end{aligned}$$

Lemma 6.3 dim *Im $F_t(C)$ is t-independent for all $|t - t_0| < \delta$ when $\delta > 0$ is sufficiently small.*

Proof of Lemma 6.3

Suppose $\{\psi_1, ..., \psi_d\}$ is a basis of Im $F_t(C)$, then since $F_{t_0}(C)e_r = e_r, 1 \leq r \leq d$, are linearly independent, and by **Theorem 6.3**, $F_t(C)$ is C^∞ differentiable in t, we have $\forall |t-t_0| < \delta, F_t(C)e_r \in$ Im $F_t(C)$ are also linearly independent, implying dim Im $F_t(C) \geq d$.

Suppose now on the contrary $\forall \delta > 0, \exists t_q$ with $|t_q - t_0| < \frac{1}{q}$ and dim Im $F_{t_q}(C) \geq d+1$. Then at least $d+1$ eigenvalues $\lambda_{h_r}(t_q), 1 \leq r \leq d+1$ lie in C^0 with orthonormal eigenvalues $\psi_{t_q h_r}$. Using the argument in the proof of **Lemma 6.2**, we may assume that after passing subsequences, $\psi_{t_0 h_r} := \lim_q \psi_{t_q h_r}$ exists and is a (p,q)-vector form with C^2 coefficients, with $\lim_q \Box_{t_q} \psi_{t_q h_r} = \Box_{t_0} \psi_{t_0 h_r}$. In both limits, convergence can be made uniform in U_j, which in turn implies $\lambda_{h_r}(t_q)$ also converges to some $\lambda_{h_r}(t_0)$. Hence,

$$\Box_{t_0} \psi_{t_0 h_r} = \lim_q \Box_{t_q} \psi_{t_q h_r} = \lim_q \lambda_{h_r}(t_q) \psi_{t_q h_r} = \lambda_{h_r}(t_0) \psi_{t_0 h_r}.$$

Since $||\psi_{t_0 h_r}||_{t_0} = \lim_q ||\psi_{t_q h_r}||_{t_q} = 1, \psi_{t_0 h_r}$ is an eigenvector with eigenvalue $\lambda_{h_r}(t_0)$. As $\lambda_{h_r}(t_q) \in C^0$ and no eigenvalue of $\Box_{t_0}$ lies on C, we have $\lambda_{h_r}(t_0) \in C^0 \Rightarrow \psi_{t_0 h_r} \in$ Im $F_{t_0}(C)$. But $(\psi_{t_0 h_r}, \psi_{t_0 h_s})_{t_0} = \lim_q (\psi_{t_q h_r}, \psi_{t_q h_s})_{t_q} = 0$ for $r \neq s$, $\psi_{t_0 h_1}, \ldots, \psi_{t_0 h_{d+1}}$ are linearly independent, implying dim Im $F_{t_0}(C) = d+1$, a contradiction. ■

Finally we come to

PROOF OF THEOREM 4.4

Let C_ε be a circle of radius $\varepsilon > 0$ centered 0 in $\mathbb{C}$ such that $C_\varepsilon \times t_0 \subseteq W$. From **Lemma 6.3**, dim Im $F_t(C_\varepsilon) =$ dim Im $F_{t_0}(C_\varepsilon)$ for all $|t-t_0| < \delta$. Since all nonzero eigenvalues of $\Box_{t_0}$ lies outside C_ε (by choosing $\varepsilon > 0$ sufficiently small), dim Im $F_{t_0}(C_\varepsilon) =$ dim Im F_{t_0}. Since the zero eigenvalues of $\Box_t$ lies inside C_ε, dim Im $F_t \leq$ dim Im $F_t(C_\varepsilon)$. Therefore, dim Im $F_t \leq$ dim Im F_{t_0}, $\forall |t-t_0| < \delta$. ■

6.1.2 The s.p.c. Compact CR Case (Theorem 5.5)

The situation becomes a bit more complicated in the CR case, for a s.p.c. CR manifold must be odd (real) dimensional, and the singled out real 1 dimensional space is somewhat different from the rest. We first clarify the differentiable structure of the family:

1. Conditions 1 and 2 of **Definition 5.10** imply that $\mathcal{M}$ is locally trivial, i.e. for all $t \in \Omega$, $M_t = M_{t_0}$ for some fixed $t_0 \in \Omega$, as differentiable manifolds. Combined with condition 3 we can conclude that $\{S_t \mid t \in \Omega\}$ is a differentiable family of subbundles of $\mathbb{C}TM_{t_0}$.

2. Let (P_t, J_t) be the real expression of S_t (c.f. **Theorem 5.1**), then $\{P_t \mid t \in \Omega\}$ is a differentiable family of contact structures on M_{t_0}. By a result of Martinet [11], $\{P_t\}$ is locally trivial and one may assume that $P_t = P_{t_0}, \forall t \in \Omega$.

Let ξ be the infinitesimal contact transformation on M_{t_0} such that $\xi \notin P_{t_0}, \forall x \in M$, and is a basic field for M_{t_0} (and for each M_t as well). Denote the corresponding basic form by θ.

We can extend J_t to a tensor field of type $(1,1)$ on M_t by setting $J_t(\xi) = 0$. Then $g_t(X,Y) = -d\theta(J_tX, Y)$ defines a tensor field g_t of type $(0,2)$ on M_t. The restriction of g_t to P_t is positive definite. We set the Riemannian metric $h_t = g_t + \theta^2$ on M_t. This induces the Hermitian inner products $< \cdot, \cdot >_t$ on $\mathbb{C}TM_t$ and $\bigwedge^k(\mathbb{C}TM_t)^*$. Now, $\forall \phi, \psi \in \mathscr{A}^k(M_t)$, define

$$(\phi, \psi)_t := \int_{M_t} < \phi, \psi >_t dv,$$

where $dv = \theta \wedge (d\theta)^{n-1}$.

Recall the complex $\{\mathscr{C}^{p,q}(M_t), d_t''\}$ defined in §5.2. The decomposition $\mathbb{C}TM_t =$

$\mathbb{C}\xi \oplus (S_t \oplus \overline{S}_t)$ is orthogonal with respect to $< \cdot, \cdot >_t$. By **Theorem 5.3**, $C^{p,q}(M_t)$ can be identified with $((\mathbb{C}\xi \oplus S_t)^*)^p \wedge (\overline{S}_t^*)^q \subseteq A^{p,q}(M_t)$. Thus, we have the orthogonal decomposition $A^{p,q}(M_t) = C^{p,q}(M_t) \oplus A^{p+1,q-1}(M_t)$, and d''_t on $\mathscr{C}^{p,q}(M_t)$ can be regarded as d on $\mathscr{A}^{p,q}(M_t)$ followed by the orthogonal projection $\mathscr{A}^{p,q+1}(M_t) \to \mathscr{C}^{p,q+1}(M_t)$. Define the formal adjoint δ''_t of d''_t and the Laplacian $\Delta''_t = d''_t\delta''_t + \delta''_t d''_t$ with respect to $(\cdot,\cdot)_t$.

By condition 3 of **Definition 5.10** again, there exists a differentiable family of bundle isomorphisms $\{\tau_t : \mathbb{C}TM_t \to \mathbb{C}TM_{t_0} \mid t \in \Omega\}$ for Ω sufficiently small, such that

$$\tau_{t_0} = \text{ id }, \qquad \tau_t S_t = S_{t_0}, \qquad \tau_t\xi = \xi$$

Each τ_t induces naturally a bundle isomorphism $C^{p,q}(M_t) \to C^{p,q}(M_{t_0})$, also denoted by τ_t.

Definition 6.3 *A family $\phi_t \in \mathscr{C}^{p,q}(M_t)$ is a differentiable family if $\tau_t\phi_t \in \mathscr{C}^{p,q}(M_{t_0})$ is differentiable in t.*

Suppose $\tau_t\phi_t \in \mathscr{C}^{p,q}(M_{t_0})$ is represented locally as $\tau_t\phi_t = \sum_{|I|=p,|J|=q} \phi^k_{tI\overline{J}}\omega^k_I \wedge \overline{\omega^k_J}$, where $\omega^k_0, ..., \omega^k_{n-1}$ are orthonormal vectors, with $\omega^k_0 = \theta$, then by writing $\omega^k_{t\,i} = \tau^{-1}\omega^k_i, 0 \le i \le n-1$ and $\overline{\omega^k_{t\,j}} = \tau^{-1}\overline{\omega^k_j}, 1 \le j \le n-1$, we have $\phi_t = \sum \phi^k_{tI\overline{J}}\omega^k_{tI}\wedge\overline{\omega^k_{tJ}}$. This implies that $\{\phi_t\}$ is a differentiable family if and only if $\phi^k_{tI\overline{J}}$ are differentiable in t. This matches with the notion of C^∞ differentiability in the case of complex manifolds.

We define the Sobolev s-norm for $s \in \mathbb{R}$ on $\mathscr{C}^{p,q}(M_t)$ as follows:

Definition 6.4 *Fix an atlas with finitely many charts (U_k, h_k) on M_{t_0} such that each U_k is homeomorphic to $\mathbb{R}^{2n-1}$. Fix a local (real) frame $(X^k_0, ..., X^k_{2n-2})$ on U_k and a partition of unity ρ_k subordinate to U_k. For all $\phi_t \in \mathscr{C}^{p,q}(M_t)$, define*

$$||\phi_t||^2_{(s)} = \sum_{I,J,k} ||(\rho_k\phi^k_{tI\overline{J}}) \circ h_k^{-1}||^2_{(s)}, \tag{6.5}$$

where $||\cdot||^2_{(s)}$ *on the RHS of* (6.5) *is the Sobolev norm on* $C_0^\infty(\mathbb{R}^{2n-1})$.

REMARK $||\phi_t||_{(s)}$ is just $||\tau_t\phi_t||_{(s)}$ on M_{t_0}, and $||\phi_t||_{(0)}$ is equivalent to $||\phi_t||_t$ uniformly in t.

From now on assume $q \neq 0, n-1$ and shrink Ω if necessary. We state the subelliptic estimates for Δ''_t essential in our proof (They are proved in **Appendix D**):

Lemma 6.4 *There exists a constant* C *such that* $\forall t \in \Omega, \forall \phi_t \in \mathscr{C}^{p,q}(M_t)$,

$$||\phi_t||^2_{(\frac{1}{2})} \leq C((\Delta''_t\phi_t, \phi_t)_t + (\phi_t, \phi_t)_t) \tag{6.6}$$

Lemma 6.5 *For any* $s = 0, 1, 2, ...$, *there is a constant* C_s *such that* $\forall t \in \Omega, \forall \phi_t \in \mathscr{C}^{p,q}(M_t)$,

$$||\phi_t||_{(\frac{s+1}{2})} \leq C_s||\Delta''_t\phi_t + \phi_t||_{(\frac{s-1}{2})} \tag{6.7}$$

Lemma 6.6 *For any* $m = 0, 1, 2, ...$, *there is a constant* c'_m *such that* $\forall t \in \Omega, \forall \phi_t \in \mathscr{C}^{p,q}(M_t)$,

$$||\phi_t||^2_{(m+1)} \leq c'_m(||\Delta''_t\phi_t||^2_{(m)} + ||\phi_t||^2_{(0)}) \tag{6.8}$$

The subelliptic estimates of the self-adjoint Δ''_t for each $t \in \Omega$ yield the strong orthogonal decomposition $\mathscr{C}^{p,q}(M_t) =$ Im $d''_t \oplus$ Im $\delta''_t \oplus \mathscr{H}^{p,q}(M_t)$, where $\mathscr{H}^{p,q}(M_t) = \ker \Delta''_t$ is finite-dimensional. This implies that $\dim H^{p,q}(M_t) = \dim \mathscr{H}^{p,q}(M_t)$. We can also obtain an orthonormal basis ψ_{th} (which are eigenvectors of Δ''_t) as in **Theorem 6.1**.

Thus, we only need to prove that $\dim \mathscr{H}^{p,q}(M_t)$ is upper semi-continuous in t in order to conclude the proof. In particular, by minor adjustment (e.g. changing $\mathscr{L}^{p,q}(M_t)$ to $\mathscr{C}^{p,q}(M_t)$, Δ to Ω and $\mathbf{H}^{p,q}(M_t)$ to $H^{p,q}(M_t)$), the arguments in §6.1.1 can be repeated for the CR case, though the "only if" argument of **Lemma 6.5** should be amended, for we can only make use of the subelliptic estimates here: We still have the inequality (6.3), but not (6.4). Instead, by repeated use of (6.8) and the equivalence of $||\cdot||_t$ and $||\cdot||_{(0)}$, we should observe that for any $\phi_t \in \mathscr{C}^{p,q}(M_t)$, $\forall m \in \mathbb{N}$,

$$||\phi_t||^2_{(m)} \leq c''_m(||\phi_t||^2_t + \sum_{l=1}^{m} ||\Delta''^l_t \phi_t||^2_t) \tag{6.9}$$

$\therefore$ For any l-th derivative $D^l_k \phi^k_{tqI\bar{J}}$ of the components of $\phi_{tq} = \sum_{h=1}^{q} b_{th}\psi_{th}$, and any compact subset K of U_k, we can choose $m \gg l$ such that for any $q > p, \forall x \in K$,

$$\begin{aligned}
&|D^l_k \phi^k_{tqI\bar{J}}(x) - D^l_k \phi^k_{tpI\bar{J}}(x)|^2 \\
&\quad \leq c_{m-l,l}||\phi_{tq} - \phi_{tp}||^2_{(m)} && (\because (6.1)) \\
&\quad \leq c_{m-l,l}c''_m\Big(||\phi_{tq} - \phi_{tp}||^2_t + \sum_{l=1}^{m} ||\Delta''^l_t(\phi_{tq} - \phi_{tp})||^2_t\Big) && (\because (6.9)) \\
&\quad \leq c_{m-l,l}c''_m \sum_{h=p+1}^{q} (1 + \lambda_h(t)^2 + \ldots + \lambda_h(t)^{2m})|b_{th}|^2,
\end{aligned}$$

which tends to 0 as $p, q \to \infty$. This suffices to conclude the proof.

6.2 Other Stability Theorems for Complex Manifolds

We now sketch the proof for **Theorem 2.3**, **Lemma 2.2**, **Theorem 4.5** and **4.6**. One shall notice that we frequently make use of the fact that $H^q(M_t, \Omega^p(B_t)) \cong \mathbf{H}^{p,q}(B_t)$ (**Appendix B**) in order to derive these theorems from the harmonic theory we have so far. We first need the following lemma:

Lemma 6.7 *If $\mathbf{H}^{p,q}(M_t)$ is t-independent, then F_t and G_t are C^∞ differentiable in t.*

PROOF OF LEMMA 6.7

With the setting in the proof of **Theorem 5.5**, we have $\forall |t-t_0|<\delta$,

$$\begin{aligned}\dim\ \mathrm{Im}\ F_t(C_\varepsilon) &= \dim\ \mathrm{Im}\ F_{t_0}(C_\varepsilon) = \dim \mathbf{H}^{p,q}(M_{t_0})\\ &= \dim \mathbf{H}^{p,q}(M_t) \qquad (\text{ By hypothesis})\\ \Rightarrow F_t(C_\varepsilon) &= F_t,\end{aligned}$$

where F_t is the orthogonal projection $\mathscr{L}^{p,q}(M_t) \to \mathbf{H}^{p,q}(M_t)$. Since $F_t(C_\varepsilon)$ is locally C^∞ differentiable in t and t_0 is arbitrary, we can concluded that F_t is C^∞ differentiable in t.

Recall that $\forall \phi_t = \sum_{h=1}^{\infty} a_{th}\psi_{th}$, $G_t\phi_t := \sum_{\lambda_h(t_0)\neq 0} \frac{a_{th}}{\lambda_h(t)}\psi_{th}$. We construct a similar function $G_t(C_\varepsilon) = \sum_{\lambda_h(t)\notin C^0} \frac{a_{th}}{\lambda_h(t)}\psi_{th}$, and it suffices to prove that $G_t = G_t(C_\varepsilon)$ locally. But this is immediate because locally $F_t(C_\varepsilon) = F_t$, implying $\lambda_h(t) = 0$ if and only if $\lambda_h(t) \in C_\varepsilon^0$. ■

We now make use of this lemma to give the following

PROOF OF THEOREM 2.3

The trick of this proof is to make use of the equality $H^1(M_t, \Theta_t) \cong \mathbf{H}^{0,1}(M_t)$. Using this equality, consider any fixed infinitesimal deformation $\frac{\partial M_t}{\partial t_m}$ obtained from $\{\theta_{jk}(t)\}$ defined on $U_{jk} = U_j \cap U_k$, and let ϕ_t be the corresponding harmonic form. Let ρ_i be the partition of unity subordinate to U_i, and define $\xi_k(t) = \sum_i \rho_i\theta_{ik}(t)$, then there exists a global $\psi_t \in \mathscr{L}^{0,1}(T(M_t))$ such that $\psi_t|_{U_j\times t} = \overline{\partial}_t\xi_j(t)$ (by using the fact that $\theta_{ik}(t)$ are holomorphic 1-cocycles). Since by definition $\psi_t = \sum_{\alpha,\beta} \frac{\partial\xi_j^\alpha(z_j,t)}{\partial \overline{z}_j^\beta} d\overline{z}_j^\beta \frac{\partial}{\partial z_j^\alpha}$ on $U_j \times t$, ψ_t is C^∞ differentiable on $t \in \Delta$.

It can be checked that $\delta^*\psi_t = \{\theta_{jk}(t)\}$, where $\delta^* : \Gamma(M_t, \overline{\partial}_t\mathscr{A}(T(M_t))) \to$

$H^1(M_t, \Theta_t)$. Since

$$\begin{aligned}\Gamma(M_t, \overline{\partial}_t \mathscr{A}(T(M_t))) &= \mathbf{H}^{0,1}(M_t) \oplus \overline{\partial}_t \mathscr{L}^{0,0}(M_t) \\ &= \mathbf{H}^{0,1}(M_t) \oplus \overline{\partial}_t \Gamma(M_t, \mathscr{A}(T(M_t))),\end{aligned}$$

$h_t F_t \psi_t = \delta^* \psi_t$ and ϕ_t are 2 elements in $\mathbf{H}^{0,1}(M_t)$ which are both mapped to $\{\theta_{jk}(t)\}$ by the isomorphism $h_t : \mathbf{H}^{p,1}(M_t) \to H^1(M_t, \Theta_t)$. Therefore, we have $F_t \psi_t = \phi_t$. Now since $\frac{\partial M_t}{\partial t_m} = 0$, we have $\phi_t = 0$ and

$$\begin{aligned}\psi_t = F_t\psi_t + \Box_t G_t \psi_t &= F_t \psi_t + \overline{\partial}_t \vartheta_t G_t \psi_t + \vartheta_t \overline{\partial}_t G_t \psi_t \\ &= F_t \psi_t + \overline{\partial}_t \vartheta_t G_t \psi_t \qquad (\because\ \overline{\partial}_t G_t \psi_t = 0) \\ &= \phi_t + \overline{\partial}_t \eta_t,\end{aligned}$$

by writing $\eta_t = \vartheta_t G_t \psi_t$. Define $\theta_j(t) = \xi_j(t) - \eta_t$, then it can be checked that $\theta_j(t)$ is holomorphic for each t, $\delta\{\theta_j(t)\} = \{\theta_{jk}(t)\}$ and their coefficients $\theta_j^\alpha(z_j, t)$ are C^∞ of (z_j, t) ($\xi_j(t)$ is C^∞ differentiable by default, and $\eta_t = \vartheta_t G_t \psi_t$ is C^∞ differentiable by **Lemma 6.7**). ■

PROOF OF THEOREM 4.5

Repeat the proof of **Theorem 2.3** as follows: For each $\frac{\partial M_t}{\partial t_k} \in H^1(M_t, \Theta_t), k = 1, ..., m$, associate $\phi_{kt} = F_t \psi_{kt} \in \mathbf{H}^{0,1}(M_t)$, which is C^∞ differentiable in t due to the hypothesis and **Lemma 6.7**. Since ϕ_{k0} correspond to the linearly independent $\left(\frac{\partial M_t}{\partial t_k}\right)_{t=0}$, ϕ_{kt} are linearly independent within $|t| < \varepsilon$, for some $\varepsilon > 0$. ■

PROOF OF LEMMA 2.2

We first prove that we can construct a basis of $H^0(M_t, \Theta_t)$ consisting of C^∞ basis vectors, and then adjust them appropriately so that they become holomorphic in all variables (in particular, holomorphic in $t_1, ..., t_m$).

Let $\{\phi_1, ..., \phi_d\}$ be a basis of $H^0(M_t, \Theta_t)$, then since the differentiable family

$\{\Theta_t \mid t \in \Delta\}$ is locally trivial, we can choose C^∞ vector fields $\psi_{tq} \in \mathscr{L}^{0,0}(M_t), q = 1, ..., d$, which are C^∞ differentiable in t and $\psi_{0q} = \phi_q$.

Note that $\mathscr{L}^{0,0}(M_t) = \mathbf{H}^{0,0}(M_t) \oplus \Box_t \mathscr{L}^{0,0}(M_t)$ and $\mathbf{H}^{0,0}(M_t) = \{\phi \in \mathscr{L}^{0,0}(M_t) \mid \overline{\partial}_t \phi = 0\} = H^0(M_t, \Theta_t)$. Now, let $\phi_{tq} = F_t \psi_{tq}$, then by the hypothesis and **Lemma 6.7**, F_t is C^∞ differentiable in t and hence so is ϕ_{tq}. By choosing $\varepsilon > 0$ small enough ϕ_{tq} are linearly independent for all $|t| < \varepsilon$, since $\phi_{0q} = \phi_q$ are assumed to be linearly independent.

Now, express such ϕ_{tq} on $U_j \times t$ as $\phi_{tq} = \sum_{\alpha=1}^{n} \phi_{qj}^\alpha(z_j, t) \dfrac{\partial}{\partial z_j^\alpha}$.

Note that the transition function $f_{jk}^\alpha(z_k, t)$ are holomorphic in all variables, and hence

$$\begin{aligned}
\frac{\partial \phi_{tq}}{\partial \bar{t}_k} &= \sum_{\alpha=1}^{n} \frac{\partial \phi_{tq}^\alpha}{\partial \bar{t}_k}(z_j, t) \frac{\partial}{\partial z_j^\alpha} = \sum_{\alpha=1} \frac{\partial}{\partial \bar{t}_k} \Big(\sum_{\beta=1}^{n} \frac{\partial f_{jk}^\alpha(z_k, t)}{\partial z_k^\beta} \phi_{qk}^\beta(z_k, t) \Big) \frac{\partial}{\partial z_j^\alpha} \\
&= \sum_{\alpha=1}^{n} \Big(\sum_{\beta=1}^{n} \frac{\partial f_{jk}^\alpha(z_k, t)}{\partial z_k^\beta} \frac{\partial \phi_{qk}^\beta(z_k, t)}{\partial \bar{t}_k} \Big) \frac{\partial}{\partial z_j^\alpha} \\
&= \sum_{\alpha=1}^{n} \frac{\partial \phi_{qj}^\alpha(z_j, t)}{\partial \bar{t}_k} \frac{\partial}{\partial z_j^\alpha}, \quad 1 \le k \le m, \; 1 \le q \le d,
\end{aligned}$$

are also holomorphic vector fields on M_t, i.e. $\dfrac{\partial \phi_{tq}}{\partial \bar{t}_k} = \sum_{p=1}^{d} a_{kpq}(t) \phi_{tp}, 1 \le q \le d$ for some C^∞ functions $a_{kpq}(t)$ of t on $|t| < \varepsilon$. We now construct $\theta_{tq} = \sum_{p=1}^{d} c_{pq}(t) \phi_{tp}$ so that θ_{tq} are linearly independent, and are holomorphic in all variables (t in particular), for some C^∞ functions $c_{pq}(t)$ of t. For such θ_{tq}, we have

$$\begin{aligned}
0 = \frac{\partial \theta_{tq}}{\partial \bar{t}_k} &= \sum_{p} \Big(\frac{\partial c_{pq}(t)}{\partial \bar{t}_k} \phi_{tp} + c_{pq}(t) \frac{\partial \phi_{tp}}{\partial \bar{t}_k} \Big) \\
&= \sum_{p} \Big(\frac{\partial c_{pq}(t)}{\partial \bar{t}_k} \phi_{tp} \Big) + \sum_{p} c_{pq}(t) \Big(\sum_{r} a_{krp}(t) \phi_{tr} \Big) \\
&= \sum_{r} \Big(\frac{\partial c_{rq}}{\partial \bar{t}_k} + \sum_{p} c_{pq}(t) a_{krp}(t) \Big) \phi_{tr}
\end{aligned}$$

$$\Rightarrow \frac{\partial c_{rq}(t)}{\partial \bar{t}_k} + \sum_p c_{pq}(t) a_{krp}(t) = 0, \qquad p, q = 1, ..., d, \qquad k = 1, ..., m$$

$$\Rightarrow \frac{\partial}{\partial \bar{t}_k} C(t) + A_k(t) C(t) = 0, \qquad k = 1, ..., m,$$

where $C(t) = (c_{rq}(t))_{r,q}$, $A_k(t) = (a_{krp}(t))_{r,p}$. We quote the following lemma (modified from Proposition 19.1 of [8]) as follows:

Lemma 6.8 *If the operators $\overline{\nabla}_k := \frac{\partial}{\partial \bar{t}_k} - A_k$ satisfy the integrability conditions: $\overline{\nabla}_k \overline{\nabla}_l - \overline{\nabla}_l \overline{\nabla}_k = 0, k, l = 1, ..., m$, then for a sufficiently small $\varepsilon > 0$, there exists a matrix function $C(t)$ of class C^∞ in t defined on $|t| < \varepsilon$ such that $\overline{\nabla}_k C(t) = 0, k = 1, ..., m$ and $C(0) = 1$.*

It can be checked that the hypothesis is satisfied from the definition of $A_k(t)$. Hence, we can choose a C^∞ $C(t)$ such that the vectors θ_{tq}'s are holomorphic (with $\theta_{0q} = \sum_p c_{pq}(0) \phi_{0p} = \sum_p \delta_{pq} \phi_{0p} = \phi_{0q}$). ■

Proof of Theorem 4.6

We refer to the orthogonal decomposition of $\mathscr{L}^{p,q}(B_t)$. From now on we fix $p \geq 0$ throughout the proof. Then it is not difficult to observe that

1. $\mathscr{L}^{p,q}(B_t) = \mathbf{H}^{p,q}(B_t) \oplus (\overline{\partial}_t \vartheta_t \mathscr{L}^{p,q}(B_t) \oplus \vartheta_t \overline{\partial}_t \mathscr{L}^{p,q}(B_t))$. The components $\mathbf{A}_t^q := \overline{\partial}_t \vartheta_t \mathscr{L}^{p,q}(B_t)$ and $\mathbf{D}_t^q := \vartheta_t \overline{\partial}_t \mathscr{L}^{p,q}(B_t)$ are orthogonal to each other.

2. $\Box_t \mathbf{A}_t^q = \mathbf{A}_t^q$, $\Box_t \mathbf{D}_t^q = \mathbf{D}_t^q$.

3. $\overline{\partial}_t : \mathbf{D}_t^{q-1} \to \mathbf{A}_t^q$ and $\vartheta_t : \mathbf{A}_t^q \to \mathbf{D}_t^{q-1}$ are bijective.

Arrange the eigenvalues of $\Box_t$ in the increasing order: $0 \leq \lambda_1^{(q)}(t) \leq ... \leq \lambda_h^{(q)}(t) \leq ...$ $(\lim_{h \to \infty} \lambda_h^{(q)}(t) = +\infty)$. Suppose we have some $u_{tk}^{q+1} \in \mathbf{A}_t^{q+1}, ||u_{tk}^{q+1}|| = 1, \Box_t u_{tk}^{q+1} = \alpha_k^{(q+1)}(t) u_{tk}^{q+1}$, then by defining $v_{tk}^q = \frac{1}{\sqrt{\alpha_k^{(q+1)}(t)}} \vartheta_t u_{tk}^{q+1}$, we have $v_{tk}^q \in \mathbf{D}_t^q, ||v_{tk}^q|| = 1$ and $\Box_t v_{tk}^q = \alpha_k^{(q+1)}(t) v_{tk}^q$. In summary, we have

Lemma 6.9 *We can choose an orthonormal basis $\{e_{t1}^q, ..., e_{td_q}^q, u_{t1}^q, ..., v_{t1}^q, ...\}$ of $\mathscr{L}^{p,q}(B_t)$ such that*

1. *$\Box_t e^q_{th} = 0$, $h = 1, \ldots, d_q$, $d_q := \dim \mathbf{H}^{p,q}(B_t)$;*

2. *There is a sequence $0 < \alpha_1^{(q)}(t) \leq \ldots \leq \alpha_k^{(q)}(t) \leq \ldots$, $\lim_{k\to\infty} \alpha_k^{(q)}(t) = +\infty$ such that $\Box_t u^q_{tk} = \alpha_k^{(q)}(t) u^q_{tk}$, $\Box_t v^q_{tk} = \alpha_k^{(q+1)}(t) v^q_{tk}$, where $u^q_t k$ and $v^q_{tk} := \frac{1}{\sqrt{\alpha_k^{(q+1)}(t)}} \vartheta_t u^{q+1}_{tk}$ are basis vectors of $\mathbf{A}^q_t$ and $\mathbf{D}^q_t$ respectively.*

3. *$\vartheta_t u^q_{tk} = \sqrt{\alpha_k^{(q)}(t)} \cdot v^{q-1}_{tk}$, $\overline{\partial}_t v^q_{tk} = \sqrt{\alpha_k^{(q+1)}(t)} \cdot u^{q+1}_{tk}$.*

Now fix an arbitrary $\varepsilon > 0$ and define

$$\begin{aligned}
h^{p,q}(t) &= \dim H^q(M_t, \Omega^p(B_t)) \; (= \dim \mathbf{H}^{p,q}(B_t)) \\
N^q(t) &= |\{\lambda_h^{(q)}(t) \mid \lambda_h^{(q)}(t) < \varepsilon, \lambda_h^{(q)}(t) \text{ is an eigenvalue of } \Box_t\}| \\
v^q(t) &= |\{\alpha_k^{(q)}(t) \mid \alpha_k^{(q)}(t) < \varepsilon\}|
\end{aligned}$$

Then, we have $N^q(t) = h^{p,q}(t) + v^q(t) + v^{q+1}(t)$, $q = 0, \ldots, n$, where we put $v^0(t) = v^{n+1}(t) = 0$.

Note that $\lambda_h^{(q)}(t)$ (q, h fixed) is continuous in $t \in \Delta$. Hence for each $t_0 \in \Delta$, $\exists \delta > 0$ such that $N^q(t) = N^q(t_0), \forall |t - t_0| < \delta$.

Since the eigenvalues are discrete in values, we can choose $\varepsilon > 0$ so that $v^q(t_0) = 0$ for $q = 1, 2, \ldots,$, then we have

$$h^{p,q}(t) + v^q(t) + v^{q+1}(t) = h^{p,q}(t_0), \qquad \forall |t - t_0| < \delta$$

Therefore, if $\dim H^{q-1}(M_t, \Omega^p(B_t))(= h^{p,q-1})$ and $\dim H^{q+1}(M_t, \Omega^p(B_t))(= h^{p,q-1})$ are t-independent, then for some $\delta > 0$, we have $v^q(t) = v^{q+1}(t) = 0$ on $|t - t_0| < \delta$, implying $h^{p,q}(t) = h^{p,q}(t_0)$, i.e. $\dim H^q(M_t, \Omega^p(B_t))$ is t-independent. Putting $\Theta_t = \mathcal{O}(T(M_t))$, the lemma is proved. ■

Appendix A

The Complex Laplacian $\Box_a$

Here we express in terms of local coordinates, the Laplacian $\Box_a$ on a holomorphic vector bundle with hermitian metric a. This computation will directly infer that $\Box_a$ is strongly elliptic.

Laplacian on M

We first deal with the $\Box$ on $C^\infty(p,q)$-forms. Suppose the hermitian metric $\sum g_{\alpha\overline{\beta}} dz^\alpha \overline{dz^\beta}$ is represented by $\sum g_{j\alpha\overline{\beta}} dz_j^\alpha \overline{dz_j^\beta}$ on U_j. Then $\omega = i \sum g_{\alpha\overline{\beta}} dz^\alpha \wedge \overline{dz^\beta}$ is the associated real $(1,1)$-form and

$$\omega^n = 2^n n! g dx^1 \wedge \ldots \wedge dx^{2n}, \qquad (x^{2\alpha-1} + ix^{2\alpha} = z^\alpha, \quad g = \det(g_{\alpha\overline{\beta}}))$$

We can express any C^∞ (p,q)-forms $\phi(z), \psi(z)$ as

$$\phi(z) = \frac{1}{p!q!} \sum \phi_{\alpha_1 \ldots \alpha_p \overline{\beta}_1 \ldots \overline{\beta}_q} dz^{\alpha_1} \wedge \ldots \wedge dz^{\alpha_p} \wedge \overline{dz^{\beta_1}} \wedge \ldots \wedge \overline{dz^{\beta_q}}$$
$$\psi(z) = \frac{1}{p!q!} \sum \psi_{\lambda_1 \ldots \lambda_p \overline{\mu}_1 \ldots \overline{\mu}_q} dz^{\lambda_1} \wedge \ldots \wedge dz^{\lambda_p} \wedge \overline{dz^{\mu_1}} \wedge \ldots \wedge \overline{dz^{\mu_q}}$$

It can be checked that

$$< \phi, \psi > (z) := \frac{1}{p!q!} \sum_{\alpha,\beta,\lambda,\mu} g^{\overline{\lambda_1}\alpha_1} \ldots g^{\overline{\beta_1}\mu_1} \ldots g^{\overline{\beta_q}\mu_q} \phi_{\alpha_1 \ldots \overline{\beta_q}}(z) \overline{\psi_{\lambda_1 \ldots \overline{\mu_q}}},$$

where $(g^{\alpha\overline{\beta}}) = (g_{\alpha\overline{\beta}})^{-1}$, is a well-defined C^∞ function of z. For convenience, we also denote

$$\psi^{\alpha_1\ldots\alpha_p\overline{\beta_1}\ldots\overline{\beta_q}}(z) := \sum_{\lambda,\mu} g^{\overline{\lambda_1}\alpha_1}\ldots g^{\overline{\beta_1}\mu_1}\ldots g^{\overline{\beta_q}\mu_q}\psi_{\lambda_1\ldots\lambda_p\overline{\mu_1}\ldots\overline{\mu_q}}(z)$$

We define the inner product by

$$(\phi,\psi) := \int_M <\phi,\psi>(z)\frac{\omega^n}{n!} = \int_M <\phi,\psi>(z)2^n g dx^1\wedge\ldots\wedge dx^{2n}$$

Define ϑ by $(\partial\overline{\phi},\psi) = (\phi,\vartheta\psi)$ for every $(p,q-1)$ form ϕ and (p,q) form ψ. The Laplacian is then given by $\Box = \overline{\partial}\vartheta + \vartheta\overline{\partial}$. We now express step by step the $\Box$ in local coordinates (In the following $A_p = (\alpha_1,\ldots,\alpha_p), \alpha_1 < \ldots < \alpha_p$):

$$\begin{aligned}
&\because (\phi,\vartheta\psi) = (\overline{\phi},\psi)\\
&= \frac{1}{q!}\int_M \sum(\overline{\partial}\phi)_{A_p\overline{\beta_0}\ldots\overline{\beta_{q-1}}}\psi^{A_p\overline{\beta_0}\ldots\overline{\beta_{q-1}}}2^n g dx^1\wedge\ldots\wedge dx^{2n}\\
&= \frac{1}{q!}\int_M \sum\left[(-1)^p\sum_{i=0}^{q-1}\overline{\partial}_{\beta_i}\phi_{A_p\overline{\beta_0}\ldots\widehat{\overline{\beta_i}}\ldots\overline{\beta_{q-1}}}\right]\psi^{A_p\overline{\beta_0}\ldots\overline{\beta_{q-1}}}2^n g dx^1\wedge\ldots\wedge dx^{2n}\\
&= -\frac{(-1)^p}{(q-1)!}\int_M \sum\phi_{A_p\overline{\beta_1}\ldots\overline{\beta_q}}\overline{\partial_\beta(\psi^{\overline{A_p}\beta\beta_1\ldots\beta_{q-1}}g)}2^n dx^1\wedge\ldots\wedge dx^{2n} \quad (\because \text{Integration by part})
\end{aligned}$$

$$\begin{aligned}
&\therefore (\vartheta\psi)^{\overline{A_p}\beta_1\ldots\beta_{q-1}}\\
&= (-1)^{p+1}\sum_{\beta=1}^{n}\left(\frac{\partial}{\partial z^\beta} + \frac{1}{g}\frac{\partial g}{\partial z^\beta}\right)\psi^{\overline{A_p}\beta\beta_1\ldots\beta_{q-1}}\\
&= (-1)^{p+1}\sum\partial_\beta\psi^{\overline{A_p}\beta\beta_1\ldots\beta_{q-1}} + \text{order 0 terms}\\
&= (-1)^{p+1}\sum g^{\overline{\alpha_1}\lambda_1}\ldots g^{\overline{\mu}\beta}g^{\overline{\mu_1}\beta_1}\ldots\partial_\beta\psi_{\lambda_1\ldots\lambda_p\overline{\mu}\overline{\mu_1}\ldots\overline{\mu_{q-1}}} + \text{order 0 terms}
\end{aligned}$$

$$\Rightarrow (\vartheta\psi)_{A_p\overline{\beta_1}\ldots\overline{\beta_{q-1}}} = (-1)^{p+1}\sum_{\beta,\mu} g^{\overline{\mu}\beta}\partial_\beta\psi_{A_p\overline{\mu}\overline{\beta_1}\ldots\beta_{q-1}} + \text{order 0 terms}$$

$$\Rightarrow (\overline{\partial}\vartheta\psi)_{A_p\overline{\beta_1}\ldots\overline{\beta_q}} = -\sum_{\beta,\mu} g^{\overline{\mu}\beta}\left[\sum_{i=1}^{q}(-1)^{i+1}\overline{\partial}_{\beta_i}\partial_\beta\psi_{A_p\overline{\mu}\overline{\beta_1}\ldots\widehat{\overline{\beta_i}}\ldots\overline{\beta_q}}\right] + \text{terms of order} \leq 1$$

On the other hand,

$$(\overline{\partial}\psi)_{A_p\overline{\beta_0}\ldots\overline{\beta_q}} = (-1)^p \sum_{i=0}^{q} (-1)^i \overline{\partial}_{\beta_i} \psi_{A_p\overline{\beta_0}\ldots\widehat{\overline{\beta_i}}\ldots\overline{\beta_q}}$$

$$\begin{aligned}(\vartheta\overline{\partial}\psi)_{A_p\overline{\beta_1}\ldots\overline{\beta_q}} &= (-1)^{p+1} \sum g^{\overline{\mu}\beta} \partial_\beta (\overline{\partial}\psi)_{A_0\overline{\mu}\overline{\beta_1}\ldots\overline{\beta_q}} + \text{terms of order } \leq 1 \\ &= -\sum_{\beta,\mu} g^{\overline{\mu}\beta} [\partial_\beta \overline{\partial}_\mu \psi_{A_p\overline{\beta_1}\overline{\beta_2}\ldots} - \partial_\beta \overline{\partial}_{\beta_1} \psi_{A_p\overline{\mu}\overline{\beta_2}\ldots} + \ldots]\end{aligned}$$

$\therefore$ After cancellation, we have

$$(\Box\psi)_{A_0\overline{\beta_1}\ldots\overline{\beta_q}} = -\sum_{\beta,\mu} g^{\overline{\mu}\beta} \partial_\beta \overline{\partial}_\mu \psi_{A_p\overline{\beta_1}\ldots\overline{\beta_q}} + \text{lower order terms}$$

REMARK There exists a real linear operator $*$ sending $C^\infty(p,q)$-forms to $C^\infty(n-p, n-q)$-forms such that $<\phi,\psi>(z)\frac{\omega^n}{n!} = \phi(z) \wedge *\overline{\psi}(z)$. Using $*$, one can conveniently write $\vartheta\psi = - * \partial(*\psi)$.

Laplacian on a Holomorphic Vector Bundle

Now, consider a holomorphic vector bundle $B^{(v)}$ over $M^{(n)}$, and let $\mathscr{A}^{p,q}(B) = \mathscr{A}(B \otimes \wedge^p T^*(M) \otimes \wedge^q \overline{T}^*(M))$. We define a complex $(\mathscr{A}^{p,q}(B), \overline{\partial})$ as follows: Suppose $\{e_{j1}(z), ..., e_{jv}(z)\}$ forms a basis of B_z for $z \in U_j$ such that on $U_j \cap U_k \neq \emptyset$,

$$e_{k\mu}(z) = \sum_{\lambda=1}^{v} f^\lambda_{jk\mu}(z) e_{j\lambda}(z).$$

Suppose also that $(z_j^1, ..., z_j^n)$ are local complex coordinates of U_j, then $\{e_{j\lambda} \otimes dz_j^I \wedge d\overline{z}^J \mid \lambda = 1, ..., v, |I| = p, |J| = q, I, J \text{ increasing}\}$ serves as a basis for $\otimes \wedge^p T^*(M) \wedge^q \overline{T}^*(M)$. With respect to this basis, each C^∞ section ϕ of $B \otimes \wedge^p T^*(M) \wedge^q \overline{T}^*(M)$ can be represented by $\phi = (\phi_j^1, ..., \phi_j^v)$ such that

$$\phi_j^\lambda = \sum_{\mu=1}^{v} f^\lambda_{jk\mu}(z) \phi_k^\mu \qquad \text{for } U_j \cap U_k \neq \emptyset$$

and $$\overline{\partial}\phi := (\overline{\partial}\phi_j^1, ..., \overline{\partial}\phi_j^v)$$

For convenience, we denote the collection of these C^∞ sections by $\Gamma(A^{p,q}(B))$.

Besides the Hermitian metric $\sum g_{\lambda\overline{\mu}} dz^\lambda d\overline{z}^\mu$ on M, we can now also define an Hermitian form on the fibres of B by specifying on each covering element U_j of M a positive definite form

$$a_j(\eta,\eta) = \sum_{\lambda,\mu} a_{j\lambda\overline{\mu}}(z)\eta_j^\lambda \overline{\eta_j^\mu}, \qquad \forall z \in U_j, \forall \eta_j \in B_z,$$

such that $a_{j\lambda\overline{\mu}}(z)$ is C^∞ and $a_j(\eta,\eta) = a_k(\xi,\xi)$ where $\xi = f_{kj}\cdot\eta$. Thus, we can define inner product for any $\phi,\psi \in \Gamma(A^{p,q}(B))$ by

$$(\phi,\psi) := \int_M <\phi,\psi> (z)\frac{\omega^n}{n!} := \int_M \sum_{\lambda,\mu=1}^{n} a_{j\lambda\overline{\mu}}(z)\phi_j^\lambda(z) \wedge *\overline{\psi_j^\mu(z)}$$

Again we define the formal adjoint ϑ_a of $\overline{\partial}$ with respect to $a = (a_{j\lambda\overline{\mu}})_{\lambda,\mu=1,\ldots,v}$ by $(\overline{\partial}\phi,\psi) = (\phi,\vartheta_a\psi), \forall\phi \in \Gamma(A^{p,q}(B)), \psi \in \Gamma(A^{p,q+1}(B))$. Then $\tau := \sum_{\lambda,\mu=1}^{v} a_{j\lambda\overline{\mu}}\phi_j^\lambda \wedge *\overline{\psi_j^\mu}$ is a $(n,n-1)$ form and hence

$$\begin{aligned}
0 = \int_M d\tau = \int_M \overline{\partial}\tau &= \int_M \sum_{\lambda,\mu} a_{j\lambda\overline{\mu}}\overline{\partial}\phi_j^\lambda \wedge *\overline{\psi_j^\mu} + (-1)^{p+q}\int_M \sum_{\lambda,\mu} \phi_j^\lambda \wedge \overline{\partial(a_{j\lambda\overline{v}} * \psi_j^v)} \\
&= \int_M \sum \phi_j^\lambda a_{j\lambda\overline{\mu}} \wedge *\overline{(\vartheta_a\psi)_j^\mu} + (-1)^{p+q}\int_M \sum \phi_j^\lambda \wedge \overline{\partial(a_{j\lambda\overline{v}} * \psi_j^v)}
\end{aligned}$$

$$\therefore \sum_\mu a_{j\mu\overline{\lambda}}(*(\vartheta_a\psi)_j^\mu) = -(-1)^{p+q}\Big(\partial(\sum_v a_{j\lambda\overline{v}} * \psi_j^v)\Big)$$

Let $(a_j^{\overline{\mu}\lambda}) = (a_{j\overline{\mu}\lambda})^{-1}$, then

$$\begin{aligned}
*(\vartheta_a\psi)_j^\mu &= -(-1)^{p+q}\sum_\lambda a_j^{\overline{\lambda}\mu}\Big(\partial(\sum_v a_{j\lambda\overline{v}} * \psi_j^v)\Big) \\
\Rightarrow \qquad (\vartheta_a\psi)_j^\mu &= -\sum_\lambda a_j^{\overline{\lambda}\mu} * \Big(\partial(\sum_v a_{j\lambda\overline{v}} * \psi_j^v)\Big) \\
&= -(*\partial * \psi)_j^\mu - \sum_{\lambda,v} a_j^{\overline{\lambda}\mu} * (\partial a_{j\lambda\overline{v}} * \psi_j^v) \\
&= (\vartheta\psi)_j^\mu + \text{terms of order zero}
\end{aligned}$$

$$\begin{aligned}
\therefore \qquad \Box_a = \overline{\partial}_a\vartheta + \vartheta\overline{\partial}_a &= \Box + \text{terms of order} \le 1 \\
&= -\sum g^{\overline{\beta}\alpha}\frac{\partial^2}{\partial z^\alpha \partial z^\beta} + \text{terms of order} \le 1 \qquad \text{(A.1)}
\end{aligned}$$

We now define strongly elliptic differential operator on any holomorphic vector bundle $B^{(v)}$. Express a self-adjoint differential operator E of even order m on $\Gamma(B)$ as

$$(E\psi)_j^{\lambda}(x) = \sum_{l=0}^{m}\sum_{\mu=1}^{v} E_{j\mu}^{l\lambda}(x, D_j)\psi_j^{\mu}(z), \quad \lambda = 1, ..., v, \forall\psi \in B_x, \forall x \in U_j,$$

where $E_{j\mu}^{l\lambda}(x, D_j)$ are homogeneous polynomials of degree l in $D_{j\alpha} = \left(\frac{\partial}{\partial z^{\alpha}}\right)_j, \alpha = 1, ..., n$. Define the *highest* order term

$$A_j(x, \xi_j, \zeta_j) = (-1)^{\frac{m}{2}} \sum_{j,\mu,\omega} a_{j\sigma\overline{\mu}}(x) E_{j\lambda}^{m\sigma}(x, \xi_j)\zeta_j^{\lambda}\overline{\zeta_j^{\mu}}, \quad \forall \xi_j \in T_x^*(M), \forall x \in M, \zeta_j \in B_x$$

Then $A(x, \xi, \zeta) := A(x, \xi_j, \zeta_j)$ on U_j is a globally defined Hermitian form in $\zeta_j^1, ..., \zeta_j^v$.

Definition A.1 *A linear partial differential operator E of even order m is said to be* ***strongly elliptic*** *if $\exists\delta > 0$ such that*

$$A(x, \xi, \zeta) \geq \delta^2|\xi|^2 \sum_{\lambda,\mu=1}^{v} a_{j\lambda\overline{\mu}}(x)\zeta_j^{\lambda}\overline{\zeta_j^{\mu}}, \quad \forall x \in M, \forall \xi \in T_x^*(M), \forall \zeta \in B_x,$$

where $\sum_{\lambda,\mu}^{v} a_{j\lambda\overline{\mu}}(x)\zeta_j^{\lambda}\overline{\zeta_j^{\mu}}$ is the Hermitian metric defined on B.

From the result of (A.1) we can easily see that $\Box_a$ is strongly elliptic.

Appendix B

Hodge-Dolbeault Theorem

Let $h : \mathscr{S} \to \mathscr{S}$" be a homomorphism of sheaves over M, define

$$\text{supp } h = \overline{\{p \in M \mid h(\mathscr{S}_p) \neq 0\}}$$

Definition B.1 *Let $\mathscr{S}$ be a sheaf over M. $\mathscr{S}$ is called a fine sheaf if for any locally finite open covering $\mathscr{U} = \{U_j\}$, there is a family of homomorphisms $h_j : \mathscr{S} \to \mathscr{S}$ such that supp $h_j \subseteq U_j$ and $\sum_j h_j =$ identity, i.e. $\sum_j h_j(s) = s, \forall s \in \mathscr{S}$.*

The sheaf of germs of C^∞ functions over M, denoted as $\mathscr{A}$, is an example of fine sheaves. The homomorphisms can be induced by the partition of unity subordinate to U_j. Meanwhile, it is not difficult to arrive at the following

Lemma B.1 *If $\mathscr{S}$ is a fine sheaf over M, then $H^q(M, \mathscr{S}) = 0, \forall q \geq 1$.*

An exact sequence of sheaves over M

$$0 \longrightarrow \mathscr{S} \xrightarrow{\iota} \mathscr{B}^0 \xrightarrow{d} \mathscr{B}^1 \xrightarrow{d} \mathscr{B}^2 \xrightarrow{d} \cdots$$

is a fine resolution of $\mathscr{S}$ if each $\mathscr{B}^i$ is fine.

Lemma B.2 *For the above fine resolution of $\mathscr{S}$, we have*

$$H^q(M, \mathscr{S}) \cong \Gamma(M, d\mathscr{B}^{q-1})/d\Gamma(M, \mathscr{B}^{q-1}), \forall q \geq 1.$$

PROOF OF LEMMA B.2

From the fine resolution, we have the following exact sequences (the underlying M is suppressed for brevity):

$$0 \longrightarrow \mathscr{S} \xrightarrow{\iota} \mathscr{B}^0 \xrightarrow{d} d\mathscr{B}^0 \longrightarrow 0$$

$$0 \longrightarrow d\mathscr{B}^{r-1} \longrightarrow \mathscr{B}^r \xrightarrow{d} d\mathscr{B}^r \longrightarrow 0,\ r = 1, 2, 3, ...$$

$$\begin{aligned}
\therefore H^1(\mathscr{S}) &\cong H^0(d\mathscr{B}^0)/dH^0(\mathscr{B}^0), & H^1(d\mathscr{B}^{r-1}) &\cong H^0(d\mathscr{B}^r)/dH^0(d\mathscr{B}^{r-1}) \\
H^q(\mathscr{S}) &\cong H^{q-1}(d\mathscr{B}^0), & H^q(d\mathscr{B}^{r-1}) &\cong H^{q-1}(d\mathscr{B}^r), \quad q = 2, 3, 4, ...
\end{aligned}$$

$$\Rightarrow H^q(\mathscr{S}) \cong H^{q-1}(d\mathscr{B}^0) \cong H^{q-1}(d\mathscr{B}^1) \cong ... \cong H^1(d\mathscr{B}^{q-2}) \cong H^0(d\mathscr{B}^{q-1})/dH^0(\mathscr{B}^{q-1}).$$

■

Considering $\mathscr{A}^{p,q}(B)$ defined in **Appendix A**, we now have the following fine resolution:

$$0 \longrightarrow \Omega^p(B) \xrightarrow{\iota} \mathscr{A}^{p,0} \xrightarrow{\overline{\partial}} \mathscr{A}^{p,1} ... \xrightarrow{\overline{\partial}} \mathscr{A}^{p,n} \longrightarrow 0,$$

where $\Omega^p(B)$ is the sheaf of germs of holomorphic F-valued p-forms. Hence, by **Lemma B.2**, we have for all $q \geq 1$,

$$H^q(M, \Omega^p(B)) \cong \Gamma(M, \overline{\partial}\mathscr{A}^{p,q-1}(B))/\overline{\partial}\Gamma(M, \mathscr{A}^{p,q-1}(B)) = H^{p,q}_{\overline{\partial}}(M, B),$$

where $H^{p,q}_{\overline{\partial}}(M, B)$ is called the $\overline{\partial}$-cohomology group of M with coefficients in B. Meanwhile, since $\Box_a$ is strongly elliptic, we have

$$\begin{aligned}
\mathscr{L}^{p,q}(B) &= \mathbf{H}^{p,q}(B) \oplus \Box_a \mathscr{L}^{p,q}(B) \\
&= \mathbf{H}^{p,q}(B) \oplus \overline{\partial}\mathscr{L}^{p,q-1}(B) \oplus \vartheta\mathscr{L}^{p,q+1}(B)
\end{aligned}$$

Observing $\Gamma(M, \overline{\partial}\mathscr{A}^{p,q-1}(B)) = \mathbf{B}^{p,q}(B) \oplus \overline{\partial}\mathscr{L}^{p,q-1}(B)$ and $\Gamma(M, \mathscr{A}^{p,q-1}(B)) \cong \mathscr{L}^{p,q-1}(B)$, we have $H^{p,q}_{\overline{\partial}}(M, B) \cong \mathbf{H}^{p,q}(B)$ and hence $H^q(M, \Omega^p(B)) \cong \mathbf{H}^{p,q}(B)$.

Appendix C

Proof of Theorem 6.2

Without loss of generality we treat all $\phi_t \in L(B_t)$ as $\phi_t \in L(B), B = B_{t_0}$, i.e. they all act on the *same* vector bundle (the parameter t actually only changes the coordinates of the bundle). We shall show, by induction on $r = 0, 1, 2, \ldots$, that in any coordinate neighbourhood U_j, if all $D^l_j \phi^\lambda_j(x,t)$ are C^r in (x,t) and $\psi_t = E_t^{-1}\phi_t$, then all $D^l_j \psi^\lambda_j(x,t)$ are also C^r in (x,t).

To achieve this, we first observe that by the assumptions on E_t and the Sobolev inequality, there exists constants such that $\forall x \in U_j, \forall l \geq 0$, there is a sufficiently large m (such that $m+1-l > n/2$) and

$$\begin{aligned} |D^l_j \phi^\lambda_j(x,t)| &\leq c^2_{m+1-l,l}||\phi_t||^2_{m+1} \leq c^2_{m+1-l,l}c'_m(||E_t\phi_t||^2_m + ||\phi_t||^2_0) \\ &\leq c^2_{m+1-l,l}c'_m(||E_t\phi_t||^2_m + \frac{1}{c}||E_t\phi_t||^2_0) \\ &\leq c_m||E_t\phi_t||^2_m, \end{aligned} \tag{C.1}$$

for some constant c_m dependent on m.

CASE I: $r = 0$

It suffices to show that any $D^l_j\psi^\lambda_j(x,t)$ is continuous in t i.e. $D^l_j\psi^\lambda_j(x,t)$ converges locally uniformly to $D^l_j\psi^\lambda_j(x,s)$ as $t \to s$. Now by (C.1), for any $l \geq 0$, $\exists m \gg l$ such that

$$\begin{aligned}|D^l_j\psi^\lambda_j(x,t) - D^l_j\psi^\lambda_j(x,s)| &\leq c_m||E_t(\psi_t - \psi_s)||_m \\ &\leq c_m||E_t\psi_t - E_s\psi_s||_m + c_m||(E_t - E_s)\psi_s||_m \\ &\leq c_m||\phi_t - \phi_s||_m + c_m||(E_t - E_s)\psi_s||_m\end{aligned}$$

$||\phi_t - \phi_s||^2_m = \sum_{l=0}^m \sum_{D^l_j} \int_{U_j} |D^l_j\phi^\lambda_j(x,t) - D^l_j\phi^\lambda_j(x,s)|^2 dX_j \to 0$ as $t \to s$, since ϕ_t is assumed to be continuous in t. Moreover, $||(E_t - E_s)\psi_s||_m \to 0$ as $(E_t\psi_s)^\lambda_j(x)$ are C^∞ functions of (x,t) (because they form a family of linear differential operators on M_{t_0} with C^∞ coefficients in (x,t).

CASE II: $r = 1$

Note that if ψ_t is C^1 differentiable in t, then

$$\eta_{kt} := \frac{\partial\psi_t}{\partial t_k} = \frac{\partial}{\partial t_k}(E_t^{-1}\phi_t) = E_t^{-1}\Big(\frac{\partial\phi_t}{\partial t_k} - \frac{\partial E_t}{\partial t_k}\phi_t\Big),$$

and hence η_{kt} is a C^0 family, because by assuming ϕ_t is C^1, ψ_t is C^0 by **CASE I**, and $\frac{\partial E_t}{\partial t_k}$ is a linear differential operator with C^∞ coefficients in (x,t). Thus, it suffices to prove that for any D^l_j, $D^l_j\psi^\lambda_j(x,t)$ is C^1 in $t_k, k = 1, ..., n$, and that $\frac{\partial}{\partial t_k}D^l_j\psi^\lambda_j(x,t) = D^l_j\eta^\lambda_{kj}(x,t)$.

Now, putting $t + h = (t_1, ..., t_{k-1}, t_k + h, t_{k+1}, ..., t_n)$,

$$\begin{aligned}&\Big|\frac{D^l_j\psi^\lambda_j(x,t+h) - D^l_j\psi^\lambda_j(x,t)}{h} - D^l_j\eta^\lambda_{kj}(x,t)\Big| \\ &\leq c_m\Big\|E_{t+h}\Big(\frac{\psi_{t+h} - \psi_t}{h} - \eta_{kt}\Big)\Big\|_m \qquad \text{for } m \gg l \text{ by (C.1)} \\ &= c_m\Big\|\Big(\frac{\phi_{t+h} - \phi_t}{h} - \frac{\partial\phi_t}{\partial t_k}\Big) - \Big(\frac{E_{t+h} - E_t}{h} - \frac{\partial E_t}{\partial t_k}\Big)\psi_t - (E_{t+h} - E_t)\eta_{kt}\Big\|_m\end{aligned}$$

$$\Big\|\frac{\phi_{t+h}-\phi_t}{h}-\frac{\partial\phi_t}{\partial t_k}\Big\|_m=\Big(\sum_{l\le m}\int\Big|\frac{D_j^l\phi_j^\lambda(x,t+h)-D_j^l\phi_j^\lambda(x,t)}{h}-\frac{\partial D_j^l\phi_j^\lambda(x,t)}{\partial t_k}\Big|^2\Big)^{\frac{1}{2}}\to 0$$

as $h\to 0$, since $D_j^l\phi_j^\lambda(x,t)$ are C^1 in (x,t). On the other hand, $\Big\|\Big(\frac{E_{t+h}-E_t}{h}-\frac{\partial E_t}{\partial t_k}\Big)\psi_t\Big\|_m$ and $||(E_{t+h}-E_t)\eta_{kt}||_m\to 0$ as $h\to 0$ since E_t is a family of linear differential operators with C^∞ coefficients in (x,t).

CASE III: $r\ge 2$

Assume the case for $r-1$ and prove the case for r. We achieve this by differentiating $\phi_t=E_t\psi_t$ with respect to t to $(r-1)$-th order:

$$\partial_t^{r-1}\phi_t=E_t\partial_t^{r-1}\psi_t+\sum_{k=0}^{r-2}(\partial_t^{r-1-k}E_t)\partial_t^k\psi_t$$

Since ϕ_t is a C^r family and ψ_t is a C^{r-1} family by assumption,

$$E_t\partial_t^{r-1}\psi_t=\partial_t^{r-1}\phi_t-\sum_{k=0}^{r-2}(\partial_t^{r-1-k}E_t)\partial_t^k\psi_t$$

is a C^1 family. By **CASE II**, $\partial_t^{r-1}\psi_t$ is also C^1. Thus, ψ_t is a C^r family. ■

Appendix D

Subelliptic Estimates of Δ_t''

In this section, we are going to derive the subelliptic estimates for Δ_t'' (**Lemma 6.4,6.5 and 6.6**). In what follows, $A \lesssim B$ means $|A| \leq C|B|$ for some constant C independent of t and $\phi_t \in \mathscr{C}^{p,q}(M_t)$ ($A \gtrsim B$ is similarly defined, and $A \sim B$ means $A \lesssim B$ and $A \gtrsim B$).

We first state the Sobolev s-norm. Let $C_0^\infty(\mathbb{R}^{2n-1})$ be the space of C^∞ functions with compact support on $\mathbb{R}^{2n-1}$. Then for all $s \in \mathbb{R}, \forall f \in C_0^\infty(\mathbb{R}^{2n-1})$,

$$||f||_{(s)}^2 = \int (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi,$$

where $\hat{f}(\xi)$ is the Fourier transform of f, i.e. $\hat{f}(\xi) = (2\pi)^{\frac{2n-1}{2}} \int e^{-i<x,\xi>} f(x)dx$.

Recall **Definition 6.4** concerning $||\phi_t||_{(s)}$ for $\phi_t \in \mathscr{C}^{p,q}(M_t)$. There are a number of basic results for $||\phi_t||_{(s)}$ quoted as follows (c.f. [3], [4], [9]):

1. **Generalized Schwarz inequality:**

$$(\phi_t, \psi_t)_t \leq ||\phi_t||_{(s)} ||\psi_t||_{(-s)}, \quad \forall s > 0, \forall \phi_t, \psi_t \in \mathscr{C}^{p,q}(M_t) \tag{D.1}$$

2. **Plancherel relation:**

$$\int |\hat{f}(\xi)|^2 d\xi = (2\pi)^{2n-1} \int |f(x)|^2 dx, \quad \forall f \in C_0^\infty(\mathbb{R}^{2n-1}) \tag{D.2}$$

Note that (D.2) implies that for all $f \in C_0^\infty(\mathbb{R}^{2n-1}), ||f||_{(s)} = ||\bigwedge^s f||$, where $\bigwedge^s f = \mathscr{F}^{-1}((1+|\xi|^2)^{s/2}\hat{f}(\xi))$, $\mathscr{F}^{-1}\hat{u}(x)$ is the inverse Fourier transform of $\hat{u}$. Similarly, we also have

$$||\phi_t||^2_{\left(\frac{s+1}{2}\right)} \sim \sum_{K,I,J} ||\bigwedge^{\frac{s}{2}}((\rho_k\phi^k_{tI\overline{J}})\circ h_k^{-1})||^2_{\left(\frac{1}{2}\right)}, \quad \forall\phi_t \in \mathscr{C}^{p,q}(M_t) \tag{D.3}$$

Now, for each fixed t, there is a unique affine connection ∇_t on M_t and a unique connection D_t on the vector bundle $\bigwedge^p(C\xi \oplus S_t)^*$. The connections induce a covariant derivative $D_{tX}\phi_t \in \mathscr{C}^{p,q}(M_t), \forall\phi_t \in \mathscr{C}^{p,q}(M_t), \forall X \in \mathbb{C}TM_t$.

Let $(e_{t_1}, ..., e_{t_{n-1}})$ be an orthonormal frame of S_t with respect to $<\cdot,\cdot>_t$. Define

$$||\phi_t||^2_{\overline{S}_t} = \int_{M_t}\sum_i < D_{t\overline{e_{t_i}}}\phi_t, D_{t\overline{e_{t_i}}}\phi_t >_t dv$$

$$||\phi_t||^2_{S_t} = \int_{M_t}\sum_i < D_{te_{t_i}}\phi_t, D_{te_{t_i}}\phi_t >_t dv$$

Then it can be proved that (ref. [15]) for $q \neq 0, n-1, \forall\phi_t \in \mathscr{C}^{p,q}(M_t)$,

$$(\Delta''_t\phi_t,\phi_t)_t = \frac{n-q-1}{n-1}||\phi_t||^2_{\overline{S}_t} + \frac{q}{n-1}||\phi_t||_{S_t} + (Q^q_t\phi_t,\phi_t)_t, \tag{D.4}$$

for some zeroth order curvature operator Q^q_t. Immediately from this we have

$$||\phi_t||^2_{\overline{S}_t} + ||\phi_t||^2_{S_t} + ||\phi_t||_t \lesssim (\Delta''_t\phi_t,\phi_t)_t + (\phi_t,\phi_t)_t \tag{D.5}$$

In addition, from the construction of the connections, we can see that they are differentiable in t such that for any differentiable family $\phi_t \in \mathscr{C}^{p,q}(M_t)$, any local complex vector field X_t on M_t,

$$(D_{tX_t}\phi_t)^k_{I\overline{J}} = X_t\phi^k_{tI\overline{J}} + \sum_{I',J'} a^{kI'\overline{J}'}_{tI\overline{J}}\phi^k_{tI'\overline{J}'},$$

where $a^{kI'\overline{J}'}_{tI\overline{J}'}$ are differentiable in t.

We now state clearly the Kohn's inequality for any differentiable famuly of sections $\phi_t \in \mathscr{C}^{p,q}(M_t)$. By Kohn's inequality on M_{t_0} (**Theorem 5.4.7** of Folland-Kohn [4], where we put the complex vector fields $A_1, ..., A_m$ to be $X_1^k, ..., X_{2n-2}^k$ and observe that the hypothesis is satisfied with $p = 2$), $\forall \phi_t \in \mathscr{C}^{p,q}(M_t)$,

$$\begin{aligned} ||\phi_t||^2_{\left(\frac{1}{2}\right)} &= \sum_{I,J,k} ||(\rho_k \phi^k_{tI\bar{J}}) \circ h_k^{-1}||_{\left(\frac{1}{2}\right)} \\ &\lesssim \sum_{I,J,k,\ 1\le l\le 2n-2} ||X_l^k(\rho_k \phi^k_{tI\bar{J}})||^2 + \sum_{I,J,k} ||\rho_k \phi^k_{tI\bar{J}}||^2 \end{aligned} \tag{D.6}$$

We can now proceed to the

Proof of Lemma 6.4

For all $\phi_t \in \mathscr{C}^{p,q}(M_t)$, we have

$$\begin{aligned} ||\phi_t||^2_{\left(\frac{1}{2}\right)} &\lesssim \sum ||X^k_{tl}(\rho_k \phi^k_{tI\bar{J}})||^2 + \sum ||\rho_k \phi^k_{tI\bar{J}}||^2 && \text{(by (D.6))} \\ &\lesssim \sum ||\rho_k (X^k_{tl} \phi^k_{tI\bar{J}})||^2 + \sum ||\phi^k_{tI\bar{J}}||^2 \\ &\lesssim \sum ||\rho_k (D_{tX^k_{tl}} \phi)^k_{tI\bar{J}}||^2 + \sum ||\phi^k_{tI\bar{J}}||^2 \\ &\lesssim ||\phi_t||^2_{S_t} + ||\phi_t||^2_{\bar{S}_t} + ||\phi_t||^2_t && \text{** See note} \\ &\lesssim (\Delta''_t \phi_t, \phi_t)_t + (\phi_t, \phi_t)_t && \text{(by (D.5))} \end{aligned}$$

**Note: This is done by using Gram Schmidt process, changing $(X^k_{t_1}, ..., X^k_{t_{n-1}})$ into a differentiable family of orthonormal frames $(e^k_{t_1}, ..., e^k_{t_{n-1}})$. ■

Proof of Lemma 6.5

We prove this lemma by induction on s. For $s = 0$, we have

$$\begin{aligned} ||\phi_t||^2_{\left(\frac{1}{2}\right)} &\lesssim (\Delta''_t \phi_t + \phi_t, \phi_t)_t && (\because \textbf{Lemma 6.4}) \\ &\lesssim ||\Delta''_t \phi_t + \phi_t||_{\left(-\frac{1}{2}\right)} ||\phi_t||_{\left(\frac{1}{2}\right)} && \text{(by (D.1))} \\ &\le \varepsilon ||\Delta''_t \phi_t + \phi_t||^2_{\left(-\frac{1}{2}\right)} + \frac{1}{4\varepsilon} ||\phi_t||^2_{\left(\frac{1}{2}\right)}, \end{aligned}$$

for any $\varepsilon > 0$ ($ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon}b^2, \forall a, b, \varepsilon > 0$). Taking $\varepsilon > 0$ to be large, we have

$$||\phi_t||^2_{\left(\frac{1}{2}\right)} \lesssim ||\Delta''_t\phi_t + \phi_t||^2_{\left(-\frac{1}{2}\right)}$$

For the general case, we need the following preparatory result (c.f. Folland-Kohn [4]):

Lemma D.1 *For any $\phi_t \in \mathscr{C}^{p,q}(M_t)$, any C^∞ function ζ_k (for each fixed k) supported in U_k with $\zeta_k \equiv 1$ on suppρ_k, let*

$$A_k\phi_t = \sum_{I,J} \zeta_k\Big(\big(\bigwedge^{\frac{s}{2}}(\rho_k\phi^k_{tI\bar{J}}) \circ h_k^{-1}\big) \circ h_k\Big)\omega^k_{tI} \wedge \overline{\omega^k_{tJ}},$$

then we have

$$||A_k\phi_t||_{(r)} \lesssim ||\phi_t||_{(r+\frac{s}{2})}, \qquad \forall r \in \mathbf{R} \tag{D.7}$$

$$(\Delta''_t A_k\phi_t + A_k\phi_t, A_k\phi_t)_t \lesssim |(A_k(\Delta''_t\phi_t + \phi_t), A_k\phi_t)_t| + ||\phi_t||^2_{\left(\frac{s}{2}\right)} \tag{D.8}$$

$\therefore$ Assuiming the case for $s-1$, we have

$$\begin{aligned}
&||\phi_t||^2_{\left(\frac{s+1}{2}\right)} \\
&\sim \sum_{k,I,J} ||\bigwedge^{\frac{s}{2}}((\rho_k\phi^k_{tI\bar{J}}) \circ h_k^{-1})||^2_{\left(\frac{1}{2}\right)} && (\because \text{(D.3)}) \\
&\sim \sum_k ||A_k\phi_t||^2_{\left(\frac{1}{2}\right)} \lesssim \sum_k (\Delta''_t A_k\phi_t + A_k\phi_t, A_k\phi_t)_t && (\because \textbf{Lemma 6.4}) \\
&\lesssim \sum_k |(A_k(\Delta''_t\phi_t + \phi_t), A_k\phi_t)_t| + ||\phi_t||^2_{\left(\frac{s}{2}\right)} && (\because \text{(D.8)}) \\
&\lesssim \sum_k ||A_k(\Delta''_t\phi_t + \phi_t)||_{\left(-\frac{1}{2}\right)} ||A_k\phi_t||_{\left(\frac{1}{2}\right)} + ||\phi_t||^2_{\left(\frac{s}{2}\right)} && (\because \text{(D.1)}) \\
&\leq \sum_k \Big(\varepsilon||A_k(\Delta''_t\phi_t + \phi_t)||^2_{\left(-\frac{1}{2}\right)} + \frac{1}{4\varepsilon}||A_k\phi_t||^2_{\left(\frac{1}{2}\right)}\Big) + ||\phi_t||^2_{\left(\frac{s}{2}\right)} \\
&\lesssim \varepsilon||\Delta''_t\phi_t + \phi_t||^2_{\left(-\frac{1}{2}+\frac{s}{2}\right)} + \frac{1}{4\varepsilon}||\phi_t||^2_{\left(\frac{1}{2}+\frac{s}{2}\right)} + ||\phi_t||^2_{\left(\frac{s}{2}\right)} && (\because \text{(D.7)}) \\
&\lesssim \varepsilon||\Delta''_t\phi_t + \phi_t||^2_{\left(\frac{s-1}{2}\right)} + \frac{1}{4\varepsilon}||\phi_t||^2_{\left(\frac{s+1}{2}\right)} + ||\Delta''_t\phi_t + \phi_t||^2_{\left(\frac{s}{2}-1\right)} && (\because \text{Induction hypothesis}) \\
&\lesssim ||\Delta''_t\phi_t + \phi_t||^2_{\left(\frac{s-1}{2}\right)},
\end{aligned}$$

by choosing $\varepsilon > 0$ sufficiently large. ■

Proof of Lemma 6.6

We proceed by induction on m. The case $m = 0$ is exactly **Lemma 6.5** when $s = 1$:

$$||\phi_t||^2_{(1)} \lesssim ||\Delta''_t \phi_t + \phi_t||^2_{(0)} \leq ||\Delta''_t \phi_t||^2_{(0)} + ||\phi_t||^2_{(0)}$$

Now, assuming the case for $m - 1$ and setting $s = 2m + 1$ in **Lemma 6.5**, we have

$$\begin{aligned}
||\phi_t||^2_{(m+1)} &\lesssim ||\Delta''_t \phi_t + \phi_t||^2_{(m)} \\
&\leq ||\Delta''_t \phi_t||^2_{(m)} + ||\phi_t||^2_{(m)} \\
&\lesssim ||\Delta''_t \phi_t||^2_{(m)} + ||\Delta''_t \phi_t||^2_{(m-1)} + ||\phi_t||^2_{(0)} \quad (\because \text{Induction hypothesis}) \\
&\lesssim ||\Delta''_t \phi_t||^2_{(m)} + ||\phi_t||^2_{(0)} \blacksquare
\end{aligned}$$

Bibliography

[1] M.S. Baouendi, P. Ebenfelt, and L.P. Rothschild, *Real submanifolds in complex space and their mappings*, Princeton University Press, Princeton NJ, 1999.

[2] A. Boggess, *CR manifolds and the tangential Cauchy-Riemann complex*, CRC Press, Boca Raton, 1991.

[3] S.C. Chen, and M.C. Shaw, *Partial differential equations in several complex variables*, Studies in Advanced Mathematics v.19, American Mathematical Society, Providence, Rhode Island, 2001.

[4] G.B. Folland, and J.J. Kohn, *The Neumann problem for the Cauchy-Riemann complex*, Princeton University Press, Princeton, NJ, 1972.

[5] D. Huybrechts, *Complex geometry: an introduction*, Springer Universitext, Springer, Berlin, 2005.

[6] K. Kodaira, *Complex manifolds and deformation of complex structures*, Springer-Verlag, New York, 1986.

[7] K. Kodaira, and J.A. Morrow: *Complex manifolds*, Holt, Rinehart and Winston, New York, 1971.

[8] K. Kodaira, and D.C. Spencer: On deformations of complex analytic structures, I-II, III, *Annals of Math.* 67 (1958), 328-466; 71(1960), 43-76.

[9] J.J. Kohn, Harmonic integrals on strongly pseudoconvex manifolds I, *Annals of Math.* 78(1963), 112-148, 79(1964), 450-472.

[10] S.G. Krantz, *Partial differential equations and complex analysis*, CRC Press, Boca Raton, 1992.

[11] J. Martinet: Forme de contact sur les variétés de dimension 3, *Proceedings of Liverpool Singularities Symposium II*, Springer-Verlag, 1971 (Lecture Notes in Math., 209, 142-163).

[12] A. Newlander, and L. Nirenberg: Complex analytic coordinates in almost complex manifolds, *Annals of Math.* 65(1957), 391-404.

[13] N. Shimakura, *Partial differential operators of elliptic type*, Translations of mathematical monographs, v.99, American Mathematical Society, Providence, Rhode Island, 1992.

[14] G. Taiani, *Cauchy-Riemann (CR) manifolds*, Monograph from Pace University Math. Dept., New York, 1989.

[15] N. Tanaka: *A differential geometric study on strongly pseudo-convex manifolds*, Kinokuniya Book-store Co., Tokyo, 1975.